Stochastic Optimal Control in Finance

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for my son,
MehmetAli'ye.

Preface

These are the extended version of the *Cattedra Galileiana* I gave in April 2003 in Scuola Normale, Pisa. I am grateful to the Society of *Amici della Scuola Normale* for the funding and to Professors Maurizio Pratelli, Marzia De Donno and Paulo Guasoni for organizing these lectures and their hospitality.

In these notes, I give a very quick introduction to stochastic optimal control and the dynamic programming approach to control. This is done through several important examples that arise in mathematical finance and economics. The theory of viscosity solutions of Crandall and Lions is also demonstrated in one example. The choice of problems is driven by my own research and the desire to illustrate the use of dynamic programming and viscosity solutions. In particular, a great emphasis is given to the problem of super-replication as it provides an usual application of these methods. Of course there are a number of other very important examples of optimal control problems arising in mathematical finance, such as passport options, American options. Omission of these examples and different methods in solving them do not reflect in any way on the importance of these problems and techniques.

Most of the original work presented here is obtained in collaboration with Professor Nizar Touzi of Paris. I would like to thank him for the fruitful collaboration, his support and friendship.

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Contents

Chapter 1

Examples and Dynamic Programming

In this Chapter, we will outline the basic structure of an optimal control problem. Then, this structure will be explained through several examples mainly from mathematical finance. Analysis and the solution to these problems will be provided later.

1.1 Optimal Control.

In very general terms, an optimal control problem consists of the following elements:

- **State process** $Z(\cdot)$. This process must capture of the minimal necessary information needed to describe the problem. Typically, $Z(t) \in \Re^d$ is influenced by the control and given the control process it has a Markovian structure. Usually its time dynamics is prescribed through an equation. We will consider only the state processes whose dynamics is described through an ordinary or a stochastic differential equation. Dynamics given by partial differential equations yield infinite dimensional problems and we will not consider those in these lecture notes.

- **Control process** $\nu(\cdot)$. We need to describe the control set, U, in which $\nu(t)$ takes values in for every t. Applications dictate the choice of U. In addition to this simple restriction $\nu(t) \in U$, there could be additional constraints placed on control process. For instance, in the stochastic setting, we will require ν to be adapted to a certain filtration, to model the flow of information. Also we may require the state process to take values in a certain region (i.e., state constraint). This also places restrictions on the process $\nu(\cdot)$.

- **Admissible controls** $\mathcal{A}$. A control process satisfying the constraints is called an admissible control. The set of all admissible controls will be denoted by $\mathcal{A}$ and it may depend on the initial value of the state process.

- **Objective functional** $J(Z(\cdot), \nu(\cdot))$. This is the functional to be maximized (or minimized). In all of our applications, J has an additive structure, or in other words J is given as an integral over time.

Then, the goal is to minimize (or maximize) the objective functional J over all *admissible controls.* The minimum value plays an important role in our analysis

$$\text{Value function: } = v = \inf_{\nu \in \mathcal{A}} J \ .$$

The main problem in optimal control is to find the minimizing control process. In our approach, we will exploit the Markovian structure of the problem and use dynamic programming. This approach yields a certain partial differential equation satisfied by the value function v. However, in solving this equation we also obtain the optimal control in a "feedback" form. This means that is the optimal process $\nu^*(t)$ is given as $\hat{\nu}(Z^*(t))$, where $\hat{\nu}$ is the optimal feedback control given as a function of the state and Z^* is the corresponding optimal state process. Both Z^* and the optimal control ν^* are computed simultaneously by solving the state dynamics with feedback control $\hat{\nu}$. Although a powerful method, it also has its technical drawbacks. This process and the technical issues will be explained by examples throughout these notes.

1.2 Examples

In this section, we formulate several important examples of optimal control problems. Their solutions will be given in later sections after the necessary techniques are developed.

1.2.1 Deterministic minimal time problem

The state dynamics is given by

$$\frac{d}{dt}Z(t) = f(Z(t), \nu(t)) \quad , t > 0 \ ,$$
$$Z(0) = z,$$

where f is a given vector field and $\nu : [0, \infty) \to U$ is the control process. We always assume that f is regular enough so that for a given control process ν, the above equation has a unique solution $Z_x^\nu(\cdot)$.

For a given target set $\mathcal{T} \subset \Re^d$, the objective functional is

$$\begin{aligned} J(Z(\cdot), \nu(\cdot)) &:= \inf\{t \geq 0 : Z_z^\nu(t) \in \mathcal{T}\} \quad \text{(or } +\infty \text{ if set is empty)}, \\ &:= T_z^\nu. \end{aligned}$$

Let

$$v(z) = \inf_{\nu \in \mathcal{A}} T_z^\nu,$$

where $\mathcal{A} := L^\infty([0, \infty); U)$, and U is a subset of a Euclidean space.

Note that additional constraints typically placed on controls. In robotics, for instance, control set U can be discrete and the state $Z(\cdot)$ may not be allowed to enter into certain a region, called obstacles.

1.2.2 Merton's optimal investment-consumption problem

This is a financial market with two assets: one risky asset, called stock, and one "riskless" asset, called bond. We model that price of the stock $S(t)$ as the solution of

$$dS(t) = S(t)[\mu dt + \sigma dW] \ , \tag{1.2.1}$$

where W is the standard one-dimensional Brownian motion, and μ and σ are given constants. We also assume a constant interest rate r for the Bond price, $B(t)$, i.e.,

$$dB(t) = B(t)[rdt] \ .$$

At time t, let $X(t)$ be the money invested in the bond, $Y(t)$ be the investments at the stock, $l(t)$ be the rate of transfer from the bond holdings to the stock, $m(t)$ be the rate of opposite transfers and $c(t)$ be the rate of consumption. So we have the following equations for $X(t)$, $Y(t)$ assuming no transaction costs.

$$dX(t) = rX(t)dt - l(t)dt + m(t)dt - c(t)dt \ , \tag{1.2.2}$$
$$dY(t) = Y(t)[\mu dt + \sigma dW] + l(t)dt - m(t)dt \ . \tag{1.2.3}$$

Set

$$Z(t) = X(t) + Y(t) = \text{wealth of the investor at time } t,$$
$$\pi(t) = \frac{Y(t)}{Z(t)} \ .$$

Then,

$$dZ(t) = Z(t)[(r + \pi(t)(\mu - r))dt + \pi(t)\sigma dW] - c(t)dt \ . \tag{1.2.4}$$

In this example, the state process is $Z = Z_z^{\pi,c}$ and the controls are $\pi(t) \in \Re^1$ and $c(t) \geq 0$. Since we can transfer funds between the stock holdings and the bond holdings instantaneously and without a loss, it is not necessary to keep track of the holdings in each asset separately.

We have an additional restriction that $Z(t) \geq 0$. Thus the set of admissible controls $\mathcal{A}_z$ is given by:

$$\mathcal{A}_z := \{(\pi(\cdot), c(\cdot)) | \text{ bounded, adapted processes so that } Z_Z^{\pi,c} \geq 0 \text{ a.s.}\} \ .$$

The objective functional is the expected discounted utility derived from consumption:

$$J = E\left[\int_0^\infty e^{-\beta t} U(c(t))dt\right] \ ,$$

where $U : [0, \infty) \rightarrow \Re^1$ is the utility function. The function $U(c) = \frac{c^p}{p}$ with $0 < p < 1$, provides an interesting class of examples. In this case,

$$v(z) := \sup_{(\pi,c)\in\mathcal{A}_z} E\left[\int_0^\infty e^{-\beta t} \ \frac{1}{p}(c(t))^p dt \mid Z_z^{\pi,c}(0) = z\right]. \tag{1.2.5}$$

The simplifying nature of this utility is that there is a certain homotethy. Note that due to the linear structure of the state equation, for any $\lambda > 0$, $(\pi, \lambda c) \in \mathcal{A}_{\lambda z}$ if and only if $(\pi, c) \in \mathcal{A}_z$. Therefore,

$$v(\lambda z) = \lambda^p v(z) \quad \Rightarrow \quad v(z) = v(1)z^p \ . \tag{1.2.6}$$

Thus, we only need to compute $v(1)$ and the optimal strategy associated to it. By dynamic programming, we will see that

$$c^*(t) = (pv(1))^{\frac{1}{p-1}} Z^*(t), \tag{1.2.7}$$

$$\pi^*(t) \equiv \pi^* = \frac{\mu - r}{\sigma^2(1-p)}. \tag{1.2.8}$$

For $v(1)$ to be finite and thus for the problem to have a solution, β needs to be sufficiently large. An exact condition is known and will be calculated by dynamic programming.

1.2.3 Finite time utility maximization

The following variant of the Merton's problem often arises in finance. Let $Z^{\pi}_{t,z}(\cdot)$ be the solution of (1.2.4) with $c \equiv 0$ and the initial condition:

$$Z^{\pi}_{t,z}(t) = z \ . \tag{1.2.9}$$

Then, for all $t < T$ and $z \in \Re^+$, consider

$$\begin{aligned} J &= E\left[\ \ U(Z^{\pi}_{t,z}(T)) \ \right], \\ v(z,t) &= \sup_{\pi \mathcal{A} t,z} E[U(Z^{\pi}_{t,z}(T)|\mathcal{F}_t], \end{aligned}$$

where $\mathcal{F}_t$ is the filtration generated by the Brownian motion. Mathematically, the main difference between this and the classical Merton problem is that the value function here depends not only on the initial value of z but also on t. In fact, one may think the pair $(t, Z(t))$ as the state variables, but in the literature this is understood only implicitly. In the classical Merton problem, the dependence on t is trivial and thus omitted.

1.2.4 Merton problem with transaction costs

This is an interesting modification of the Merton's problem due to Constantinides [9] and Davis & Norman [12]. We assume that whenever we move funds from bond to stock we pay, or loose, $\lambda \in (0,1)$ fraction to the transaction fee ,and similarly, we loose $\mu \in (0,1)$ fraction in the opposite transfers. Then, equations (1.2.2),(1.2.3) become

$$dX(t) = rX(t)dt - l(t)dt + (1-\mu)m(t)dt - c(t)dt \ , \tag{1.2.10}$$

$$dY(t) = Y(t)[\mu dt + \sigma dW] + (1-\lambda)l(t)dt - m(t)dt \ . \tag{1.2.11}$$

In this model, it is intuitively clear that the variable $Z = X + Y$, is not sufficient to describe the state of the model. So, it is now necessary to consider the pair $Z := (X, Y)$ as the state process. The controls are the processes l, m and c, and all are assumed to be non-negative. Again,

$$v(x,y) := \sup_{\nu=(l,m,c)\in\mathcal{A}_{x,y}} E\left[\int_0^\infty e^{-\beta t}\frac{1}{p}(c(t))^p dt\right] .$$

The set of admissible controls are such that the solutions $(X_x^\nu, Y_x^\nu) \in \mathcal{L}$ for all $t \geq 0$. The liquidity set $\mathcal{L} \subset \Re^2$ is the collection of all (x, y) that can be transferred to a non-negative position both in bond and stock by an appropriate transaction, i.e.,

$$\begin{aligned}\mathcal{L} &= \{(x,y)\in\Re^2 \quad : \quad \exists(L,M)\geq 0 \text{ s.t.} \\ &\qquad (x+(1-\mu)M-L, Y-M+(1-\lambda)L)\in\Re^+\times\Re^+\} \\ &= \{(x,y)\in\Re^2 \quad : \quad (1-\lambda)x+y\geq 0 \text{ and } x+(1-\mu)y\geq 0\} .\end{aligned}$$

An important feature of this problem is that it is possibly *singular*, i.e., the optimal $(l(\cdot), m(\cdot))$ process can be unbounded. On the other hand, the nonlinear penalization $c(t)^p$ does not allow $c(t)$ to be unbounded.

The singular problems share this common feature that the control enters linearly in the state equation and either is not included in the objective functional or included only in a linear manner.

So, it is convenient to introduce processes:

$$L(t) := \int_0^t l(s)ds, \qquad M(t) := \int_0^t m(s)ds .$$

Then, $(L(\cdot), M(\cdot))$ are nondecreasing adopted processes and $(dL(t), dM(t))$ can be defined as random measures on $[0, \infty)$. With this notation, we rewrite (1.2.10), (1.2.11) as

$$\begin{aligned} dX &= rXdt - dL + (1-\mu)dM - c(t)dt , \\ dY &= Y[\mu dt + \sigma dW] + (1-\lambda)dL - dM , \end{aligned}$$

and $\nu = (L, M, c) \in \mathcal{A}_{x,y}$ is admissible if they are adapted (L, M) nondecreasing, $c \geq 0$ and

$$(X_x^\nu(t), Y_y^\nu(t)) \in \mathcal{L} \quad \forall t \geq 0 . \tag{1.2.12}$$

1.2.5 Super-replication with portfolio constraints

Let $Z^{\pi}_{t,z}(\cdot)$ be the solution of (1.2.4) with $c \equiv 0$ and (1.2.9), and let $S_{t,s}(\cdot)$ be the solution of (1.2.1) with $S_{t,s}(t) = s$. Given a deterministic function $G : \Re^1 \to \Re^1$ we wish to find

$$v(t,s) := \inf\{z \mid \exists\pi(\cdot) \text{ adapted, } \pi(t) \in K \text{ and } Z^{\pi}_{t,z}(T) \geq G(S_{t,s}(T)) \text{ a.s. } \} ,$$

where T is maturity, K is an interval containing 0, i.e., $K = [-a, b]$. Here a is related to a short-sell constraint and b to a borrowing constraint (or equivalently a constraint on short-selling the bond).

This is clearly not in the form of the previous problems, but it can be transferred into that form. Indeed, set

$$\mathcal{X}(z,s) := \begin{cases} 0, & z \geq G(s), \\ +\infty, & z < G(s). \end{cases}$$

Consider an objective functional,

$$J(t,s,s;\pi(\cdot)) := E\left[\mathcal{X}(Z^{\pi}_{t,z}(T), S_{t,s}(T)) \mid \mathcal{F}_t \right] ,$$

$$u(t,z,s) := \inf_{\pi \in \mathcal{A}} J(t,s,s;\pi(\cdot)) ,$$

and $\pi \in \mathcal{A}$ if and only if π is adapted with values in K. Then, observe that

$$u(t,z,s) = \begin{cases} 0, & z > v(t,s) \\ +\infty, & z < v(t,s) . \end{cases}$$

and at $z = v(t,z,s)$ is a subtle question. In other words,

$$v(t,s) = inf\{z | \, u(t,z,s) = 0\} .$$

1.2.6 Buyer's price and the no-arbitrage interval

In the previous subsection, we considered the problem from the perspective of the writer of the option. For a potential buyer, if the quoted price z of a certain claim is low, there is a different possibility of arbitrage. She would take advantage of a low price by buying the option for a price z. She would finance this purchase by using the instruments in the market. Then she tries to maximize her wealth (or minimize her debt) with initial wealth of $-\,z$. If at maturity,

$$Z^{\pi}_{t,-z}(T) + G(S_{t,s}(T)) \geq 0, \quad a.s.,$$

then this provides arbitrage. Hence the largest of these initial data provides the lower bound of all prices that do not allow arbitrage. So we define (after observing that $Z^{\pi}_{t,-z}(T) = -Z^{\pi}_{t,z}(T)$),

$$\underline{v}(t,s) := \sup\{z \mid \exists \pi(\cdot) \text{ adapted, } \pi(t) \in K \text{ and } Z^{\pi}_{t,z}(T) \le G(S_{t,s}(T)) \text{ a.s. } \} \ .$$

Then, the *no-arbitrage interval* is given by

$$[\underline{v}(t,s), v(t,s)] \ .$$

In the presence of friction, there are many approaches to pricing. However, the above above interval must contain all the prices obtained by any method.

1.2.7 Super-replication with gamma constraints

To simplify, we take $r = 0$, $\mu = 0$. We rewrite (1.2.4) as

$$dZ(t) = n(t)dS(t) \ ,$$

$$dS(t) = S(t)\sigma dW(t) \ .$$

Then, $n(t) = \pi(t)Z(t)/S(t)$ is the number of stocks held at a given time. Previously, we placed no restrictions on the time change of rate of $n(\cdot)$ and assumed only that it is bounded and adapted. The gamma constraint, restricts $n(\cdot)$ to be a semimartingale,

$$dn(t) = dA(t) + \gamma(t)dS(t) \ ,$$

where A is an adapted BV process, $\gamma(\cdot)$ is an adapted process with values in an internal $[\gamma_*, \gamma^*]$.

Then, the super-replication problem is

$$v(t,s) := \inf\{z \mid \exists \nu = (n(t), A(\cdot), \gamma(\cdot)) \in \mathcal{A}_{t,s,z} \text{ s.t. } Z^{\nu}_{t,z}(T) \ge G(S_{t,s}(T))\} \ .$$

The important new feature here is the singular form of the equation for the $n(\cdot)$ process. Notice the dA term in that equation.

1.3 Dynamic Programming Principle

In this section, we formulate an abstract dynamic programming following the recent manuscript of Soner & Touzi [20]. This principle holds for all dynamic optimization problems with a certain structure. Thus, the structure of the problem is of critical importance. We formulate this in the following main assumptions.

Assumption 1 *We assume that for every control ν and initial data (t,z), the corresponding state process starts afresh at every stopping time $\tau > t$, i.e.,*

$$Z^{\nu}_{t,z}(s) = Z^{\nu}_{\tau, Z^{\nu}_{t,z}(\tau)}(s) \ , \qquad \forall s \le \tau \ .$$

Assumption 2 *The affect of ν is causal, i.e., if $\nu^1(s) = \nu^2(s)$ for all $s \le \tau$, where τ is a stopping time, then*

$$Z^{\nu^1}_{t,z}(s) = Z^{\nu^2}_{t,z}(s) \ , \qquad \forall s \le \tau \ .$$

Moreover, we assume that if ν is admissible at (t,z) then, ν is restricted to the stochastic interval $[\tau, T]$ is also admissible starting at $(\tau, Z^{\nu}_{t,z}(\tau))$.

Assumption 3 *We also assume that the concatenation of admissible controls yield another admissible control. Mathematically, for a stopping time τ and $\nu \in \mathcal{A}_{t,z}$, set $\eta^{\nu} = (\tau, Z^{\nu}_{t,z}(\tau))$. Suppose $\hat{\nu} \in \mathcal{A}_{\eta^{\nu}}$ and define*

$$\overline{\nu}(s) = \begin{cases} \nu(s), & s \le \tau \ , \\ \hat{\nu}(s), & s \ge \tau \ . \end{cases}$$

Then, we assume $\overline{\nu} \in \mathcal{A}_{t,z}$. Precise formulation is in Soner & Touzi [20].

Assumption 4 *Finally, we assume an additive structure for J, i.e.,*

$$J = \int_t^{\tau} L(s, \nu(s), Z^{\nu}_{t,z}(s))ds + G(Z^{\nu}_{t,z}(\tau)) \ .$$

The above list of assumptions need to be verified in each example. Under these structural assumptions, we have the following result which is called the *dynamic programming principle* or DPP in short.

Theorem 1.3.1 (Dynamic Programming Principle) *For any stopping time* $\tau \geq t$

$$v(t,z) = \inf_{\nu \in \mathcal{A}_{t,z}} E\left[\int_t^\tau L ds + v(\tau, Z^\nu_{t,z}(\tau)) \mid \mathcal{F}_t \right] .$$

We refer to Fleming & Soner [14] and Soner & Touzi [20]for precise statements and proofs.

1.3.1 Formal Proof of DPP

By the additive structure of the cost functional,

$$\begin{aligned} v(t,z) &= \inf_{\nu \in \mathcal{A}_{t,z}} E(\int_t^T L ds + G(Z^\nu_{t,z}(T)) \mid \mathcal{F}_t) \\ &= \inf_{\nu \in \mathcal{A}_{t,z}} E(\int_t^\tau L ds \ + \ E[\int_\tau^T L ds + G(Z^\nu_{t,z}(T)) \mid \mathcal{F}_\tau] \mid \mathcal{F}_t) \end{aligned} \tag{1.3.13}$$

By Assumption 2, ν restricted to the interval $[\tau, T]$ is in $\mathcal{A}_{\tau, Z^\nu_{t,z}(\tau)}$. Hence,

$$E[\int_t^T L ds + G \mid \mathcal{F}_\tau] \geq v(\eta^\nu),$$

where

$$\eta^\nu := \eta^\nu_{t,z}(\tau) = (\tau, Z^\nu_{t,z}(\tau)) .$$

Substitute the above inequality into (1.3.13) to obtain

$$v(t,z) \geq \inf_\nu E[\int_t^\tau L ds + v(\eta^\nu) \mid \mathcal{F}_t] .$$

To prove the reverse inequality, for $\varepsilon > 0$ and $\omega \in \Omega$, choose $\nu_{\omega,\varepsilon} \in \mathcal{A}_{\eta^\nu}$ so that

$$E(\int_\tau^T L(s, \nu_{\omega,\varepsilon}(s), Z^{\nu_{\omega,\varepsilon}}_{\eta^\nu}(s)) ds + G(Z^{\nu_{\omega,\varepsilon}}_{\eta^\nu}(T)) | \mathcal{F}_\tau) \leq v(\eta^\nu) + \varepsilon .$$

For a given $\nu \in \mathcal{A}_{t,z}$, set

$$\nu^*(s) := \begin{cases} \nu(s), & s \in [t, \tau] \\ \nu_{\omega,\varepsilon}(s), & \tau \leq s \leq T \end{cases}$$

Here there are serious measurability questions (c.f. Soner & Touzi), but it can be shown that $\nu^* \in \mathcal{A}_{t,z}$. Then, with $Z^* = Z^{\nu^*}_{t,z}$,

$$\begin{aligned} v(t,z) &\leq E\Big(\int_t^T L(s,\nu^*(s),Z^*(s))ds + G(Z^*(T)) \mid \mathcal{F}_t\Big) \\ &\leq E\Big(\int_t^\tau L(s,\nu(s),Z(s))ds + v(\eta^\nu) + \varepsilon \mid \mathcal{F}_t\Big) \ . \end{aligned}$$

Since this holds for any $\nu \in \mathcal{A}_{t,z}$ and $\varepsilon > 0$,

$$v(t,z) \leq \inf_{\nu \in \mathcal{A}_{t,z}} E\Big(\int_\tau^T L ds + v(\eta^\nu) + |\mathcal{F}_t\Big) \ .$$

□

The above calculation is the main trust of a rigorous proof. But there are technical details that need to provided. We refer to the book of Fleming & Soner and the manuscript by Soner & Touzi.

1.3.2 Examples for the DPP

Our assumptions include all the examples given above. In this section we look at the super-replication, and more generally a target reachability problem and deduce a geometric DPP from the above DPP. Then, we will outline an example for theoretical economics for which our method does that always apply.

Target Reachability.

Let $Z^\nu_{t,z}, \mathcal{A}_{t,z}$ be as before. Given a target set $\mathcal{T} \subset \Re^d$. Consider

$$V(t) := \{z \in \Re^d \ : \ \exists \nu \in \mathcal{A}_{t,z} \text{ s.t. } Z^\nu_{t,z}(T) \in \mathcal{T} \text{ a.s.}\} \ .$$

This is the generalization of the super-replication problems considered before. So as before, for $A \subset \Re^d$, set

$$\mathcal{X}_A(z) := \begin{cases} 0, & z \in \mathcal{T} \ , \\ +\infty & z \notin \mathcal{T} \ , \end{cases}$$

and

$$v(t,z) := \inf_{\nu \in \mathcal{A}_{z,t}} E[\mathcal{X}_\mathcal{T}(Z^\nu_{t,z}(T)) \mid \mathcal{F}_t] \ .$$

Then,

$$v(t,z) = \mathcal{X}_{V(t)}(z) = \left\{ \begin{array}{cc} 0, & z \in V(t) \ , \\ +\infty, & z \notin V(t) \ . \end{array} \right.$$

Since, at least formally, DPP applies to v,

$$\begin{aligned} v(t,z) = \mathcal{X}_{V(t)}(z) &= \inf_{\nu \in \mathcal{A}_{t,z}} E(v(\tau, Z^{\nu}_{t,z}(\tau)) \mid \mathcal{F}_t) \\ &= \inf_{\nu \in \mathcal{A}_{t,z}} E(\mathcal{X}_{V(\tau)}(Z^{\nu}_{t,z}(\tau)) \mid \mathcal{F}_t) \ . \end{aligned}$$

Therefore, $V(t)$ also satisfies a geometric DPP:

$$V(t) = \{ z \in \Re^d \ : \ \exists \nu \in \mathcal{A}_{t,z} \text{ s.t. } Z^{\nu}_{t,z}(\tau) \in V(\tau) \text{ a.s. } \} \ . \tag{1.3.14}$$

In conclusion, this is a nonstandard example of dynamic programming, in which the principle has the above geometric form. Later in these notes, we will show that this yields a geometric equation for the time evolution of the reachability sets.

Incentive Controls.

Here we describe a problem in which the dynamic programming does not always hold. The original problem of Benhabib [4] is a resource allocation problem. Two players are using $y(t)$ by consuming $c_i(t), i = 1,2$. The equation for the resource is

$$\frac{dy}{dt} = \eta(y(t)) - c_1(t) - c_2(t) \ ,$$

with the constraint

$$y(t) \geq 0, \quad c_i \geq 0 \ .$$

If at some time t_0, $y(t_0) = 0$, after this point we require $y(t), c_i(t) = 0$ for $t \geq t_0$. Each player is trying to maximize

$$v_i(y) := \sup_{c_i} \int_0^{\infty} e^{-\beta t} c_i(t) dt \ .$$

Set

$$v(y) := \sup_{c_1 + c_2 = c} \int_0^{\infty} e^{-\beta t} c(t) dt \ ,$$

so that, clearly, $v_1(y) + v_2(y) \leq v(y)$. However, each player may bring the state to a bankruptcy by consuming a large amount to the detriment of the other player and possibly to herself as well.

To avoid this problem Rustichini [17] proposed a variation in which the state equation is

$$\frac{d}{dt}X(t) = f(X(t), c(t)) \ ,$$

with initial condition

$$X(0) = x \ .$$

Then, the pay-off is

$$J(x, c(\cdot)) = \int_0^\infty e^{-\beta t} L(t, X(t), c(t)) dt \ ,$$

and $c(\cdot) \in \mathcal{A}_x$ if

$$\int_t^\infty e^{-\beta s} \ L(t, X(t), c(t)) ds \geq e^{-\beta s} D(X_x^c(t), c(t)), \qquad \forall t \geq 0 \ ,$$

where D is a given function. Note that this condition, in general, violates the concatenation property of the set of admissible controls. Hence, dynamic programming does not always hold. However, Barucci-Gozzi-Swiech [3] overcome this in certain cases.

1.4 Dynamic Programming Equation

This equation is the infinitesimal version of the dynamic programming principle. It is used, generally, in the following two ways:

- Derive the DPE *formally* as we will do later in these notes.
- Obtain a smooth solution, or show that there is a smooth solution via PDE techniques.
- Show that the smooth solution is the value function by the use of Ito's formula. This step is called the *verification*.
- As a by product, an optimal policy is obtained in the verification.

This is the classical use of the DPE and details are given in the book Fleming & Soner and we will outline it in detail for the Merton problem.

The second approach is this:

- Derive the DPE rigorously using the theory of viscosity solutions of Crandall and Lions.
- Show uniqueness or more generally a comparison result between sub and super viscosity solutions.
- This provides a unique characterization of the value function which can then be used to obtain further results.

This approach, which become available by the theory of viscosity solutions, avoids showing the smoothness of the value function. This is very desirable as the value function is often not smooth.

1.4.1 Formal Derivation of the DPE

To simplify the presentation, we only consider the state processes which are diffusions. Let the state variable X be the unique solution

$$dX = \mu(t, X(t), \nu(t))dt + \sigma(t, X(t), \nu(t)))dW \ ,$$

and a usual pay-off functional

$$J(t, x, \nu) = E[\int_t^T L(s, X^{\nu}_{t,x}(s), \nu(s))ds + G(X^{\nu}_{t,x}(T)) \mid \mathcal{F}_t] \ ,$$

$$v(t, x) := \inf_{\nu \in \mathcal{A}_{t,x}} J(t, x, \nu) \ .$$

We assume enough so that the DPP holds. Use $\tau = t + h$ in the DPP to obtain

$$v(t, x) = \inf_{\nu \in \mathcal{A}_{t,x}} E[\int_t^{t+h} Lds + v(t + h, X^{\nu}_{t,x}(t + h)) \mid \mathcal{F}_t] \ .$$

We assume, *without justification*, that v is sufficiently smooth. This part of the derivation is *formal* and can not be made rigorous unless viscosity theory is revoked. Then, by the Ito's formula,

$$v(t + h, X^{\nu}_{t,x}(t + h)) = v(t, x) + \int_t^{t+h} (\frac{\partial}{\partial t} v + \mathcal{L}^{\nu(s)} v)ds + \text{ martingale} \ ,$$

where

$$\mathcal{L}^{\nu} v := \mu(t, x, \nu) \cdot \nabla v + \frac{1}{2} tr\, a(t, x, \nu) D^2 v \ ,$$

with the notation,

$$a(t,x,\nu) := \sigma(t,x,\nu)\sigma(t,x,\nu)^t \quad \text{and} \quad tr\ a := \sum_{i=1}^{d} a_{ii}\ .$$

In view of the DPP,

$$\sup_{\nu\in\mathcal{A}_{t,x}} E[-\int_t^{t+h} (\frac{\partial}{\partial t} v + \mathcal{L}^{\nu(s)} v + L) ds] = 0\ .$$

We assume that the coefficients μ, a, L are continuous. Divide the above equation by h and let h go to zero to obtain

$$-\frac{\partial}{\partial t} v(t,x) + H(x,t,\nabla v(t,x), D^2 v(t,x)) = 0\ , \tag{1.4.15}$$

where

$$H(x,t,\xi,A) := \sup\{-\mu.\xi - \frac{1}{2} tr\, aA - l \quad : \quad (\mu,a,l) \in A(t,x)\}\ ,$$

and $(\mu,a,l) \in A(t,x)$ iff there exists $\nu \in \mathcal{A}_{t,x}$ such that

$$(\mu,a,l) = \lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} (\mu(x,\nu(s),t), a(x,\nu(s),t), L(x,\nu(s),t)) ds\ .$$

We should emphasize that we assume that the functions μ, a, L are sufficiently regular and we also made an unjustified assumption that v is smooth. All these assumptions are not needed in the theory of viscosity solutions as we will see later or we refer to the book by Fleming & Soner.

For a large class of problems, $\mathcal{A}_{t,x} = L^\infty((0,\infty)\times\Omega; U)$ for some set U. Then,

$$\mathcal{A}(t,x) = \{(\mu(x,\nu,t), a(x,\nu,t), L(x,\nu,t) \quad : \quad \nu \in U\}\ ,$$

and

$$H(x,t,\xi,A) = \sup_{\nu\in U}\{-\mu(x,\nu,t)\cdot\xi - \frac{1}{2} tr\, a(x,\nu,t)A - L(x,\nu,t)\}\ .$$

1.4.2 Infinite horizon

A important class of problems are known as the discounted infinite horizon problems. In these problems the state equation for X is time homogenous and the time horizon $T = \infty$. However, to ensure the finiteness of the cost functional, the running cost is exponentially discounted, i.e.,

$$J(x, \nu) := E \int_0^\infty e^{-\beta t} L(s, X_x^\nu(s), \nu(s)) ds \ .$$

Then, following the same calculation as in the finite horizon case, we derive the dynamic programming equation to be

$$\beta v(x) + H(x, \nabla v(x), D^2 v(x)) = 0 \ , \tag{1.4.16}$$

where for $\mathcal{A}_{t,x} = L^\infty((0,\infty) \times \Omega; U)$,

$$H(x, \xi, A) = \sup_{\nu \in U} \{ -\mu(x, \nu) \cdot \xi - \frac{1}{2} tr\, a(x, \nu) A - L(x, \nu) \} \ .$$

1.5 Examples for the DPE

In this section, we will obtain the corresponding dynamic programming equation for the examples given earlier.

1.5.1 Merton Problem

We already showed that, in the case with no transaction costs,

$$v(z) = v(1) z^p \ .$$

This is a rare example of an interesting stochastic optimal control problem with a smooth and an explicit solution. Hence, we will employ the first of the two approaches mentioned earlier for using the DPE.

We start with the DPE (1.4.16), which takes the following form for this equation (accounting for sup instead of inf),

$$\beta v(z) + \inf_{\pi \in \Re^1, c \geq 0} \{ -(r + \pi(\mu - r)) z v_z(z) - \frac{1}{2} \pi^2 \sigma^2 z^2 v_{zz} + c v_z - \frac{1}{p} c^p \} = 0 \ .$$

We write this as,

$$\beta v(z) - rzv_z(z) - \sup_{\pi \in \Re^1} \{\pi(\mu - r)zv_z + \frac{1}{2}\pi^2\sigma^2 z^2 v_{zz}\} - \sup_{c\geq 0}\{-cv_z + \frac{1}{p}c^p\} = 0 \ .$$

We directly calculate that (for $v_z(z) > 0 > v_{zz}(z)$)

$$\beta v(z) - rzv_z(z) - \frac{1}{2}\frac{((\mu - r)zv_z(z))^2}{\sigma^2 z^2 v_{zz}(z)} - H(v_z(z)) = 0 \ ,$$

where

$$H(v_z(z)) = \frac{1-p}{p}(v_z(z))^{\frac{p}{p-1}} \ ,$$

with maximizers

$$\pi^* = -\frac{(\mu - r)zv_z(z)}{\sigma^2 z^2 v_{zz}(z)}, \qquad c^* = (v_z(z))^{\frac{1}{p-1}} \ .$$

We plug the form $v(z) = v(1)z^p$ in the above equations. The result is the equation (1.2.7) and (1.2.8) and

$$v(1)\left[\beta - rp - \frac{p(\mu - r)^2}{2(1-p)\sigma^2}\right] - \frac{1-p}{p}(p\ v(1))^{\frac{p}{p-1}} = 0 \ .$$

The solution is

$$v(1) = \alpha := \frac{(1-p)^{1-p}}{p}\ \left[\beta - rp - \frac{p(\mu - r)^2}{2(1-p)\sigma^2}\right]^{p-1} \ ,$$

and we require that

$$\beta > rp + \frac{p(\mu - r)^2}{2(1-p)\sigma^2} \ .$$

Although the above calculations look to be complete, we recall that the derivation of the DPE is formal. For that reason, we need to complete these calculations with a verification step.

Theorem 1.5.1 (Verification) *The function αz^p, with α as above, is the value function. Moreover, the optimal feedback policies are given by the equations* (1.2.7) *and* (1.2.8).

Proof. Set $u(z) := \alpha z^p$.

For $z > 0$ and $T > 0$, let $\nu = (\pi(\cdot), c(\cdot)) \in \mathcal{A}_z$ be any admissible consumption, investment strategy. Set $Z := Z_z^\nu$. Apply the Ito's rule to the function $e^{-\beta\ t}\ u(Z(t))$. The result is

$$e^{-\beta T} E[u(Z(T))] = u(z) + \int_0^T\ e^{-\beta t}\ E[-\beta\ u(Z(t)) + \mathcal{L}^{\pi(t),c(t)} u(Z(t))]\ dt,$$

where $\mathcal{L}^{\pi,c}$ is the infinitesimal generator of the wealth process. By the fact that u solves the DPE, we have, for any π and c,

$$\beta\ u(z) - \mathcal{L}^{\pi,c} u(z) - \frac{1}{p} c^p \geq 0.$$

Hence,

$$u(z) \geq E[e^{-\beta\ T}\ u(Z(T)) + \int_0^T\ e^{-\beta\ t}\ \frac{1}{p}(c(t))^p\ dt]\ .$$

By direct calculations, we can show that

$$\lim_{T\to\infty}\ E[e^{-\beta\ T}\ u(Z(T))] = \lim_{T\to\infty}\ E[e^{-\beta\ T}\ \alpha\ (Z(T))^p] = 0\ .$$

Also by the Fatou's Lemma,

$$\lim_{T\to\infty}\ E[\int_0^T\ e^{-\beta\ t}\ \frac{1}{p}(c(t)^p\ dt] = J(z; \pi(\cdot), c(\cdot)).$$

Since this holds for any control, we proved that

$$u(z) = \alpha z^p \geq v(z) = \text{value function}\ .$$

To prove the opposite inequality we use the controls (π^*, c^*) given by the equations (1.2.7) and (1.2.8). Let Z^* be the corresponding state process. Then,

$$\beta\ u(z) - \mathcal{L}^{\pi^*,c^*} u(z) - \frac{1}{p}(c^*)^p = 0\ .$$

Therefore,

$$\begin{aligned} Ee^{-\beta T} u(Z^*(T)) &= u(z) + \int_0^T\ e^{-\beta t} E[-\beta\ u(Z^*(t)) + \mathcal{L}^{\pi^*(t),c^*(t)} u(Z^*(t))] dt \\ &= u(z) + E[\int_0^T\ e^{-\beta\ t}\ \frac{1}{p}\ (c^*(t))^p\ dt]\ . \end{aligned}$$

Again we let T tend to infinity. Since Z^* and the other quantities can be calculated explicitly, it is straightforward to pass to the limit in the above equation. The result is

$$u(z) = J(z; \pi^*, c^*).$$

Hence, $u(z) = v(z) = J(z; \pi^*, c^*)$. □

1.5.2 Minimal Time Problem

For $\mathcal{A}_x = L^\infty((0,\infty); U)$, the dynamic programming equation has a simple form,

$$\sup_{\nu \in U} \{-f(x,\nu) \cdot \nabla v(x) - 1\} = 0 \qquad x \notin \mathcal{T} \ .$$

This follows from our results and the simple observation that

$$J = \tau_x^\nu = \int_0^{\tau_x^\nu} 1 ds \ .$$

So we may think of this problem an infinite horizon problem with zero discount, $\beta = 0$.

In the special example, $U = B_1$, $f(x,\nu) = \nu$, the above equation simplifies to the Eikonal equation,

$$|\nabla v(x)| = 1 \qquad x \notin \mathcal{T} \ ,$$

together with the boundary condition,

$$v(x) \ = 0, \qquad x \in \mathcal{T} \ .$$

The solution is the distance function,

$$v(x) = \inf_{y \in \mathcal{T}} \{|x - y|\} = |x - y^*| \ ,$$

and the optimal control is

$$\nu^* = \frac{y^* - x}{|y^* - x|} \qquad \forall t \ .$$

As in the Merton problem, the solution is again explicitly available. However, $v(x)$ is not a smooth function, and the above assertions have to be proved by the viscosity theory.

1.5.3 Transaction costs

Using the formulation (1.2.10) and (1.2.11), we formally obtain,

$$\begin{aligned}\beta v(x,y) \quad + \quad & \inf_{(l,m,c)\geq 0}\{-rxv_x - \mu y v_y - \frac{1}{2}\sigma^2 y^2 v_{yy} - \\ & l[(1-\lambda)v_y - v_x] - m[(1-\mu)v_x - v_y] + cv_x - \frac{1}{p}c^p\} = 0 \ .\end{aligned}$$

We see that, since l and m can be arbitrarily large,

$$(1-\lambda)v_y - v_x \leq 0 \text{ and } (1-\mu)v_x - v_y \leq 0 \ . \tag{1.5.17}$$

Also dropping the l and m terms we have,

$$\beta v - rxv_x - \mu y v_y - \frac{1}{2}\sigma^2 y^2 v_{yy} := \beta v - \mathcal{L}v \geq \mathcal{H}(v_x) \ ,$$

where

$$\mathcal{H}(v_x) := \sup_{c\geq 0} \left\{\frac{1}{p}c^p - cv_x\right\} \ .$$

Moreover, if both inequalities are strict in (1.5.17), then the above is an equality. Hence,

$$\min\{\beta v - \mathcal{L}v - \mathcal{H}(v_x), v_x - (1-\lambda)v_y, v_y - (1-\mu)v_x\} = 0 \ . \tag{1.5.18}$$

This derivation is extremely formal, but can be verified by the theory of viscosity solutions, cf. Fleming & Soner [14], Shreve & Soner [18]. We also refer to Davis & Norman [12] who was first to study this problem using the first approach described earlier.

Notice also the singular character of the problem resulted in a quasi variational inequality, instead of a more standard second order elliptic equation.

We again use homothety $\lambda^p v(x,y) = v(\lambda x, \lambda y)$, to represent $v(x,y)$ so

$$v(x,y) = (x+y)^p f(\frac{y}{x+y}), \qquad\qquad (x,y) \in \mathcal{L} \ ,$$

where

$$f(u) := v(1-u,u), \qquad -\frac{1-\lambda}{\lambda} \leq u \leq \frac{1}{\mu} \ .$$

The DPE for v, namely (1.5.18), turns into an equation for f the coefficient function; a one dimensional problem which can be solved numerically by standard methods.

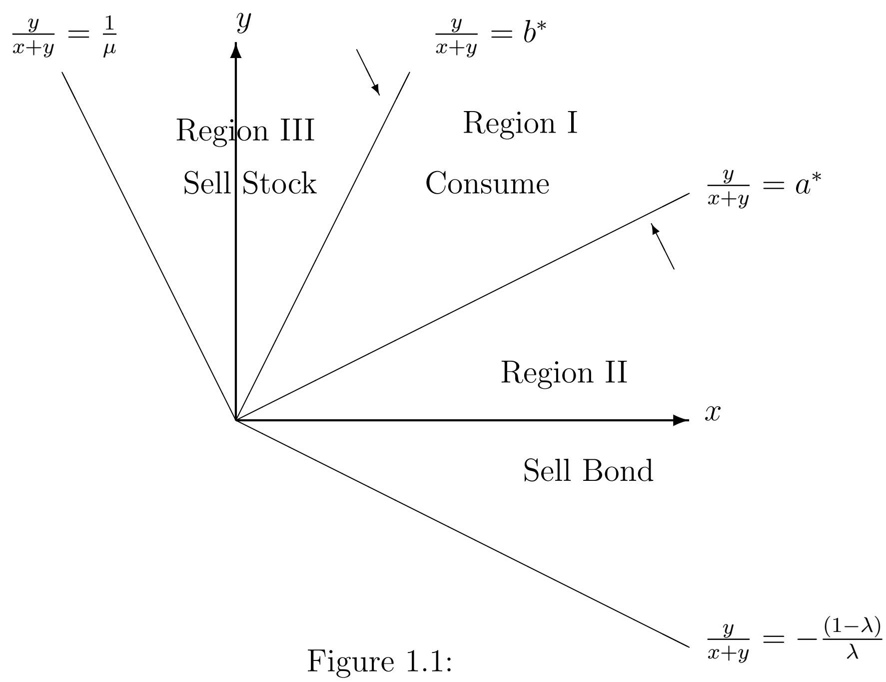

Figure 1.1:

Further, we know that v is concave. Using the concavity of v, we show that there are points
$-\frac{(1-\lambda)}{\lambda} \leq a^* < \pi^*_{mert} < b^* \leq \frac{1}{\mu}$ so that

$$\text{In Region I:} \quad \beta v - \mathcal{L}v - \mathcal{H}(v_x) = 0 \quad \text{for} \quad a^* \leq \frac{y}{x+y} \leq b^* \ ,$$

$$\text{In Region II:} \quad v_x - (1-\lambda)v_y = 0 \quad \text{for} \quad -\frac{(1-\lambda)}{\lambda} \leq \frac{y}{x+y} \leq a^* \ ,$$

$$\text{In Region III:} \quad v_y - (1-\mu)v_x = 0 \quad \text{for} \quad b^* \leq \frac{y}{x+y} \leq \frac{1}{\mu} \ .$$

So we formally expect that in Region 1, no transactions are made and consumption is according to , $c^* = (v_x(x,y))^{\frac{1}{p-1}}$. In Region 2, we sell bonds and buy stocks, and in Region 3 we sell stock and buy bonds. Finally, the process $(X(t), Y(t))$ is kept in Region 1 through reflection.

Constant b^*, a^* are not explicitly available but can be computed numerically.

1.5.4 Super-replication with portfolio constraints

In the next Chapter, we will show that the DPE is (with $K = (-a, b)$)

$$\min\{-\frac{\partial v}{\partial t} - \frac{1}{2}s^2\sigma^2 v_{ss} - rsv_s + rv \quad ; bv - sv_s; \quad sv_s + av\} = 0$$

with the final condition

$$u(T, s) = G(s) \ .$$

Then, we will show that

$$u(t, s) = E^*[\hat{G}(S_{t,s}(T)) \mid \mathcal{F}_t] \ ,$$

where E^* is the risk neutral expectation and $\hat{G}$ is the minimal function satisfying

(i). $\hat{G} \geq G$,

(ii) $-a \leq \frac{s\hat{G}_s(s)}{\hat{G}_s} \leq b$.

Examples of $\hat{G}$ will be computed in the last Chapter.

1.5.5 Target Reachability Problem

Consider

$$dX = \mu(t, z, \nu(t))dt + \sigma(t, z, \nu(t))dW \ ,$$

as before, $\mathcal{A} = L^\infty((0, \infty); U)$. The reachability set is given by,

$$V(t) := \{x | \ \exists \nu \in \mathcal{A} \text{ such that } X^\nu_{t,x}(T) \in \mathcal{T} \text{ a.s}\} \ ,$$

where $\mathcal{T} \subset \Re^d$ is a given set. Then, at least formally,

$$v(t, x) := \mathcal{X}_{V(t)}(x) = \lim_{\varepsilon \downarrow 0} H^\varepsilon(w(t, x)) \ ,$$

where

$$w(t, x) = \inf_{\nu \in \mathcal{A}} E[G(X^\nu_{t,x}(T)) \mid \mathcal{F}_t] \ ,$$

$$H^\varepsilon(w) = \frac{\tanh(w/\varepsilon) + 1}{2} \ ,$$

and $G : \Re^d \to [0, \infty)$ any smooth function which vanishes only on $\mathcal{T}$, i.e., $G(x) = 0$ if and only if $x \in \mathcal{T}$. Then, the standard DPE yields,

$$-\frac{\partial w}{\partial t} + \sup_{\nu \in U}\{-\mu(t, x, \nu) \cdot \nabla w - \frac{1}{2}tr(a(t, x, \nu)D^2 w)\} = 0 \ .$$

We calculate that, with $w^\varepsilon := H^\varepsilon(w)$,

$$\frac{\partial w^\varepsilon}{\partial t} = (H^\varepsilon)'\frac{\partial w}{\partial t}, \quad \nabla w^\varepsilon = (H^\varepsilon)'\nabla w, \quad D^2 w^\varepsilon = (H^\varepsilon)' D^2 w + (H^\varepsilon)''\nabla w \otimes \nabla w \ .$$

Hence,

$$(H^\varepsilon)' D^2 w = D^2 w^\varepsilon - \frac{(H^\varepsilon)''}{[(H^\varepsilon)']^2}\nabla w^\varepsilon \otimes \nabla w^\varepsilon \ .$$

This yields,

$$-\frac{\partial w^\varepsilon}{\partial t} + \sup_{\nu\in U}\{-\mu\cdot\nabla w^\varepsilon - \frac{1}{2} tr\, a D^2 w^\varepsilon + \frac{1}{2}\frac{(H^\varepsilon)''}{[(H^\varepsilon)']^2}\, a\nabla w^\varepsilon\cdot\nabla w^\varepsilon\} = 0 \ .$$

Here, *very formally*, we conjecture the following limiting equation for $v = \lim w^\epsilon$:

$$-\frac{\partial w}{\partial t} \sup_{\nu\in\mathcal{K}(\nabla w)}\{-\mu\cdot\nabla w - \frac{1}{2} tr\, a D^2 w\} = 0 \ . \tag{1.5.19}$$

$$\mathcal{K}(\xi) := \{\nu \quad : \quad \sigma^t(t,x,\nu)\xi = 0\} \ .$$

Notice that for $\nu \in \mathcal{K}(\nabla w^\varepsilon)$, $a\nabla w^\varepsilon\cdot\nabla w^\varepsilon = 0$. And if $\nu \notin \mathcal{K}(\nabla w^\varepsilon)$ then, $a\nabla w^\varepsilon\cdot\nabla w^\varepsilon = |\sigma^t\nabla w^\varepsilon|^2 > 0$ and $(H^\varepsilon)''/[(H^\varepsilon)']^2 \approx 1/H$ will cause the nonlinear term to blow-up.

The above calculation is very formal. A rigorous derivation using the viscosity solution and different methods is available in Soner & Touzi [20, 21].

Mean Curvature Flow.

This is an interesting example of a target reachability problem, which provides a stochastic representation for a geometric flow. Indeed, consider a target reachability problem with a general target set and state dynamics

$$dX = \sqrt{2}(I - \nu\otimes\nu)dW \ ,$$

where $\nu(\cdot) \in \partial B_1$ is a unit vector in $\Re^d$. In our previous notation, the control set U is set of all unit vectors, and $\mathcal{A}$ is the collection of all adapted process with values in U. Then, the geometric dynamic programming equation (1.5.19) takes the form

$$-\frac{\partial v}{\partial t} + \sup_{\nu\in K(\nabla v)}\{-tr(I-\nu\otimes\nu)D^2 v\} = 0 \ ,$$

and

$$\mathcal{K}(\nabla v) = \{\nu \in B_1 \quad : \quad (I - \nu \otimes \nu)\nabla v = 0\} = \{\pm \frac{\nabla v}{|\nabla v|}\} \ .$$

So the equation is

$$-\frac{\partial v}{\partial t} - \Delta v + \frac{D^2 v \nabla v \cdot \nabla v}{|\nabla v|^2} = 0 \ .$$

This is exactly the level set equation for the mean curvature flow as in the work of Evans-Spruck [13] and Chen-Giga-Goto [6].

If we use

$$dX = \sqrt{2}\Pi(t) dW \ ,$$

where $\Pi(t)$ is a projection matrix on $\Re^d$ onto $(d-k)$ dimensional planes then we obtain the co-dimension k mean curvature flow equation as in Ambrosio & Soner [1].

Chapter 2

Super-Replication under portfolio constraints

In this Chapter, we will provide all the technical details for this specific problem as an interesting example of a stochastic optimal control problem.

For this problem, two approaches are available. In the first, after a clever duality argument, this problem is transformed into a standard optimal control problem and then solved by dynamic programming, we refer to Karatzas & Shreve [15] for details of this method. In the second approach, dynamic programming is used directly. Although, when available the first approach provides more insight, it is not always possible to apply the dual method. The second approach is a direct one and applicable to all super-replication problems. The problem with Gamma constraint is an example for which the dual method is not yet known.

2.1 Solution by Duality

Let us recall the problem briefly. We consider a market with one stock and one bond. By multiplying $e^{r(T-t)}$ we may take $r = 0$, (or equivalently taking the bond as the numeriare). Also by a Girsanov transformation, we may take $\mu = 0$. So the resulting simpler equations for the stock price and wealth processes are

$$dS(t) = \sigma S(t) dW(t) \ ,$$

$$dZ(t) = \sigma \pi(t) Z(t) dW(t) \ .$$

A contingent claim with payoff $G : [0,\infty) \to \Re^1$ is given. The minimal super-replication cost is

$$v(t,s) = \inf\{z \quad : \quad \exists \pi(\cdot) \in \mathcal{A} \text{ s.t. } Z^{\pi}_{t,z}(T) \geq G(S_{t,s}(T)) \text{ a.s. } \} ,$$

where $\mathcal{A}$ is the set of all essentially bounded, adapted processes $\pi(\cdot)$ with values in a convex set K.

This restriction of $\pi(\cdot) \in K$, corresponds to proportional borrowing (or equivalently short-selling of bond) and short-selling of stock constraints that the investors typically face.

The above is the so-called writer's price. The buyer's point of view is slightly different. The appropriate target problem for this case is

$$\hat{v}(t,s) := \sup\{z \quad : \quad \exists \pi(\cdot) \in \mathcal{A} \text{ s.t. } Z^{\pi}_{t,-z}(T) + G(S_{t,s}(T)) \geq 0 \text{ a.s. } \} .$$

Then, the interval $[\hat{v}(t,s), v(t,s)]$ gives the no-arbitrage interval. That is, if the initial price of this claim is in this interval, then, there is no admissible portfolio process $\pi(\cdot)$ which will result in a positive position with probability one.

2.1.1 Black-Scholes Case

Let us start with the unconstrained case, $K = \Re^1$. Since $Z^{\pi}_{t,z}(\cdot)$ is a martingale, if there is z and $\pi(\cdot) \in \mathcal{A}$ which is super-replicating, then

$$z = Z^{\nu}_{t,z}(t) = E[Z^{\pi}_{t,z}(T) \mid \mathcal{F}_t] \geq E[G(S_{t,s}(T) \mid \mathcal{F}_t] .$$

Our claim is, indeed the above inequality is an equality for $z = v(t,s)$. Set

$$Y_u := E[G(S_{t,s}(T)) \mid \mathcal{F}_u] .$$

By the martingale representation theorem, $Y(\cdot)$ is a stochastic integral. We choose to write it as

$$Y(u) = E[G(S_{t,s}(T))|\mathcal{F}_t] + \int_t^u \sigma \pi^*(\rho) Y(\rho) dW(\rho) ,$$

with an appropriate $\pi^*(\cdot) \in \mathcal{A}$. Then,

$$Y(\cdot) = Z^{\pi^*}_{t,z_0}(\cdot), \quad z_0 = E[G(S_{t,s}(T)) \mid \mathcal{F}_t] .$$

Hence, $v(t,s) \geq z_0$. But we have already shown that if an initial capital supports a super-replicating portfolio then, it must be larger than z_0. Hence,

$$v(t,s) = z_0 = E[G(S_{t,s}(T)) \mid \mathcal{F}_t] := v^{BS}(t,s) \ ,$$

which is the Black-Scholes price. Note that in this case, starting with z_0, there always exists a replicating portfolio.

In this example, it can be shown that the buyer's price is also equal to the Black-Scholes price v^{BS}. Hence, the no-arbitrage interval defined in the previous subsection is the singleton $\{v^{BS}\}$. Thus, that is the only fair price.

2.1.2 General Case

Let us first introduce several elementary facts from convex analysis. Set

$$\delta_K(\nu) := \sup_{\pi \in K} -\pi v, \quad \tilde{K} := \{\nu : \quad \delta_K(\nu) < \infty\} \ .$$

In the convex analysis literature, δ_K is the support function of the convex set K. In one dimension, we may directly calculate these functions. However, we use this notation, as it is suggestive of the multidimensional case. Then, it is a classical fact that

$$\pi \in K \Leftrightarrow \pi\nu + \delta_K(\nu) \geq 0 \quad \forall \nu \in \tilde{K} \ .$$

Let $z, \pi(\cdot)$ be an initial capital, and respectively, a super-replicating portfolio. For any $\nu(\cdot)$ with values in $\tilde{K}$, let P^ν be such that

$$W^\nu(u) := W(u) + \int_t^u \nu(\rho) \frac{1}{\sigma} d\rho$$

is a P^ν martingale. This measure exists under integrability conditions on $\nu(\cdot)$, by the Girsanov theorem. Here we assume essentially bounded processes, so P^ν exits. Set

$$\tilde{Z}(u) := Z^\pi_{t,z}(u) \ \exp(-\int_t^u \delta_K(\nu(\rho)) d\rho) \quad .$$

By calculus,

$$d\tilde{Z}(u) = \tilde{Z}(u)[-(\delta_K(\nu(u)) + \pi(u)\nu(u))du + \sigma dW^\nu(u)] \ .$$

Since $\pi(u) \in K$ and $\nu(u) \in \tilde{K}$, $\delta_K(\nu(u) + \pi(u)\nu(u) \geq 0$ for all u. Therefore, $\tilde{Z}(u)$ is a super-martingale and

$$E^\nu[\tilde{Z}(T) \mid \mathcal{F}_t] \leq \tilde{Z}(t) = Z^\pi_{t,z}(t) = z \ .$$

Also $Z^\pi_{t,z}(T) \geq G(S_{t,s}(T))$ $P-$a.s, and therefore, P^ν-a.s. as well. Hence,

$$\begin{aligned} \tilde{Z}(T) &= \exp(-\int_t^T \delta_K(\nu(v))du)\ Z^\pi_{t,z}(T) \\ &\geq exp(-\int_t^T \delta_K(\nu(v))du)\, G(S_{t,s}(T)) \quad P^\nu - \text{a.s.}\ . \end{aligned}$$

All of these together yield,

$$v(t,s) \geq z^\nu := E^\nu[exp(-\int_t^T \delta_K(\nu(v))du)G(S_{t,s}(T)) \mid \mathcal{F}_t] \ .$$

Since this holds for any $\nu(\cdot) \in \tilde{K}$,

$$v(t,s) \geq \sup_{\nu \in \tilde{K}} z^\nu \ .$$

The equality is obtained through a super-martingale representation for the right hand side, c.f. Karatzas & Shreve [15]. The final result is

Theorem 2.1.1 (Cvitanic & Karatzas [11]) *The minimal super replicating cost $v(t,s)$ is the value function of the standard optimal control problem,*

$$v(t,s) \ = \ E\left[exp(-\int_t^T \delta_K(\nu(v))du)\ G(S^\nu_{t,s}(T)) \mid \mathcal{F}_t\right] \ ,$$

where $S^\nu_{t,s}$ solve

$$dS^\nu_{t,s} = S^\nu_{t,s}(T)\ [-\nu dt + \sigma dW] \ .$$

Now this problem can be solved by dynamic programming. Indeed, an explicit solution was obtained by Broadie, Cvitanic & Soner [5]. We will obtain this solution by the direct approach in the next section.

2.2 Direct Solution

In this section, we will use dynamic programming directly to obtain a solution. The dynamic programming principle for this problem is

$$v(t,s) = \inf\{Z \quad : \quad \exists \pi(\cdot) \in \mathcal{A} \text{ s.t. } Z^{\pi}_{t,z}(\tau) \geq v(\tau, S_{t,s}(\tau)) \text{ a.s. } \} \ .$$

We will use this to derive first the dynamic programming equation and then the solution.

2.2.1 Viscosity Solutions

We refer to the books by Barles [2], Fleming & Soner [14], the User's Guide [10] for an introduction to viscosity solutions and for more references to the subject.

Here we briefly introduce the definition. For a locally bounded function v, set

$$v^*(t,s) := \limsup_{(t',s')\to(t,s)} v(t,s), \qquad v_*(t,s) := \liminf_{(t',s')\to(t,s)} v(t,s) \ .$$

Consider the partial differential equation,

$$F(t,s,v,v_t,v_s,v_{ss}) = 0 \ .$$

We say that v is a viscosity *supersolution* if for any $\varphi \in C^{1,2}$ and any *minimizer* (t_0,s_0) of $(u_* - \varphi)$,

$$F(t_0,s_0,u_*(t_0,s_0),\varphi_t(t_0,s_0),\varphi_s(t_0,s_0),\varphi_{ss}(t_0,s_0)) \geq 0 \ . \tag{2.2.1}$$

A subsolution satisfies

$$F(t_0,s_0,u^*(t_0,s_0),\varphi_t,\varphi_s,\varphi_{ss}) \leq 0 \ , \tag{2.2.2}$$

at any *maximizer* of $(u^* - \varphi)$.

Note that a viscosity solution of $F = 0$ is not a viscosity solution of $-F = 0$.

It can be checked that the distance function introduced in the minimal time problem is a viscosity solution of the Eikonal equation.

Theorem 2.2.1 *The minimal super-replicating cost v is a viscosity solution of the DPE,*

$$\min\{-\frac{\partial v}{\partial t} - \frac{1}{2}s^2\sigma^2 v_{ss} - rsv_s + rv \quad ; bv - sv_s; \quad sv_s + av\} = 0 \ . \tag{2.2.3}$$

The proof will be given in the next two subsections.

2.2.2 Supersolution

Assume $G \geq 0$. Let $\varphi \in C^{1,2}$ and

$$v_*(t_0, s_0) - \varphi(t_0, s_0) = 0 \leq (v_* - \varphi)(t, s) \qquad \forall (t, s).$$

Choose t_n, s_n, z_n such that

$$(t_n, s_n) \to (t_0, s_0), \qquad v(t_n, s_n) \to v_*(t_0, s_0),$$

$$v(t_n, s_n) \leq z_n \leq \varphi(t_n, s_n) + \frac{1}{n^2}.$$

Then, by the dynamic programming principle, there is $\pi^n(\cdot) \in \mathcal{A}$ so that

$$Z^n(t_n + \frac{1}{n}) \geq v(t_n + \frac{1}{n}, S^n(t_n + \frac{1}{n})) \text{ a.s } ,$$

where

$$Z^n := Z^{\pi^n}_{t_n, z_n}, \qquad S^n := S_{t_n, s_n}.$$

Since $\varphi \leq v_* \leq v$,

$$Z^n(t_n + \frac{1}{n}) \geq \varphi(t_n + \frac{1}{n}, S^n(t_n + \frac{1}{n})) \text{ a.s } .$$

We use the Ito's rule and the dynamics of $Z^n(\cdot)$ to obtain,

$$\begin{aligned} z_n + \int_{t_n}^{t_n + \frac{1}{n}} \sigma \pi^n(u) Z^n(u) dW(u) \quad \geq \quad & \varphi(t_n, s_n) \\ & + \int_{t_n}^{t_n + \frac{1}{n}} (\varphi_t + \mathcal{L}\varphi)(u, S^n(u)) du \\ & + \int_{t_n}^{t_n + \frac{1}{n}} \sigma \varphi_s(u, S^n(u)) S^n(u) dW(u) \ . \end{aligned}$$

We rewrite this as

$$c_n + \int_{t_n}^{t_n + \frac{1}{n}} a_n(u) du + \int_{t_n}^{t_n + \frac{1}{n}} b_n(u) dW(u) \geq 0 \text{ a.s. } , \forall n \ ,$$

where

$$c_n := z_n - \varphi(t_n, s_n) \in [0, \frac{1}{n^2}],$$

$$a_n(u) = -(\varphi_t + \mathcal{L}\varphi)(u, S^n(u)), \quad [\mathcal{L}\varphi = \frac{1}{2}\sigma^2 s^2 \varphi_{ss}],$$

$$b_n(u) = \sigma\ [\pi^n(u)Z^n(u) - \varphi_s(u, S^n(u))S^n(u)]\ .$$

For a real number $\lambda > 0$, let $P^{\lambda,n}$ be so that

$$W_n^\lambda(u) := W(u) + \int_t^u b_n(\rho)d\rho$$

is a $P^{\lambda,n}$ martingale. Set

$$\begin{aligned} M^n(u) &:= c_n + \int_{t_n}^u a_n(\rho)d\rho + \int_{t_n}^u b_n(\rho)dW(\rho) \\ &= c_n + \int_{t_n}^u (a_n(\rho) - \lambda b_n^2(\rho))d\rho + \int_{t_n}^u b_n(\rho)dW^{\lambda,n}(\rho)\ . \end{aligned}$$

Since $M^n(u) \geq 0$,

$$\begin{aligned} 0 &\leq E^{\lambda,n} M^n(t_n + \frac{1}{n}) \\ &= c_n + E^{\lambda,n}[\int_{t_n}^{t_n+\frac{1}{n}} (a_n(\rho) - \lambda b_n^2(\rho))\ d\rho]\ . \end{aligned}$$

Note that $a_n(\rho) \to -(\varphi_t - \mathcal{L}\varphi)(t_0, s_0)$ as $\rho \to t_0$. We multiply the above inequality n and let n tend to infinity. The result is for every $\lambda > 0$,

$$(\varphi_t - \mathcal{L}\varphi)(t_0, s_0) \leq -\lambda \liminf_{n\to\infty} E^{\lambda,n} n \int_{t_n}^{t_n+\frac{1}{n}} \sigma^2(\pi^n(u)v_*(t_0, s_0) - s_0\varphi_s(t_0, s_0))^2 du.$$

Hence,

$$-(\varphi_t - \mathcal{L}\varphi)(t_0, s_0) \geq 0\ .$$

$$\liminf_{n\to\infty} E^{\lambda,n} n \int_{t_n}^{t_n+\frac{1}{n}} \sigma^2(\pi^n(u)v_*(t_0, s_0) - s_0\varphi_s(t_0, s_0))^2 du = 0\ .$$

Moreover, since $v_* \geq 0$ and $\pi_n(\cdot) \in (-a, b]$,

$$b\ v_*(t_0, s_0) - s_0\varphi_s(t_0, s_0) \geq 0,$$

$$s_0\varphi_s(t_0, s_0) + a\ v_*(t_0, s_0) \geq 0. \tag{2.2.4}$$

In conclusion,

$$F(t_0, s_0, u_*(t_0, s_0), \varphi_t(t_0, s_0), \varphi_s(t_0, s_0), \varphi_{ss}(t_0, s_0)) \geq 0, \qquad (2.2.5)$$

where

$$F(t, a, v, q, \xi, A) = \min\{-q - \frac{1}{2}\sigma^2 s^2 A;\; bv - s\xi;\; av + s\xi\}.$$

Here q stands for φ_t, ξ for φ_s and A for φ_{ss}.

□

Thus, we proved that

Theorem 2.2.2 *v is a viscosity super solution of*

$$F(t, s, v, v_t, v_s, v_{ss}) = \min\{v_t - \frac{\sigma^2}{2} s^2 v_{ss};\; bv - sv_s;\; av + sv_s\} \geq 0,$$

on $(0, T) \times (0, \infty)$.

2.2.3 Subsolution

Assume that $G \geq 0$ and $G \not\equiv 0$. Let $\varphi \in C^{1,2}, (t_0, s_0)$ be such that

$$(v^* - \varphi)(t_0, s_0) = 0 \geq (v^* - \varphi)(t, s) \qquad \forall (t, s)\ .$$

By considering $\hat{\varphi} = \varphi + (t - t_0)^2 + (s - s_0)^4$ we may assume the above maximum of $(v^* - \varphi)$ is strict. (Note $\hat{\varphi}_t = \varphi_t$, $\hat{\varphi}_s = \varphi_s$, $\hat{\varphi}_{ss} = \varphi_{ss}$ at (t_0, s_0).) We need to show that

$$F(t_0, s_0, v^*(t_0, s_0), \varphi_t(t_0, s_0), \varphi_s(t_0, s_0), \varphi_{ss}(t_0, s_0) \leq 0\ . \qquad (2.2.6)$$

Suppose to the contrary. Since $v^* = \varphi$ at (t_0, s_0), and since F and φ are smooth, there exists a neighborhood of (t_0, s_0), say $\mathcal{N}$, and $\delta > 0$ so that

$$F(t, s, \varphi, \varphi_t, \varphi_s, \varphi_{ss}) \geq \delta \qquad \forall (t, s) \in \mathcal{N}\ . \qquad (2.2.7)$$

Since $G \geq$ and $G \not\equiv 0$, and $s_0 \neq 0$ then $v^*(t_0, s_0) > 0$. So $\varphi > 0$ in a neighborhood of (t_0, s_0). Also since $(v^* - \varphi)$ has a strict maximum at (t_0, s_0), there is a subset of $\mathcal{N}$, denoted by $\mathcal{N}$ again so that

$$\varphi > 0 \text{ on } \overline{\mathcal{N}}, \quad \text{and} \quad v^*(t, s) \leq e^{-\delta}\varphi(t, s) \quad \forall (t, s) \notin \mathcal{N}\ .$$

Set
$$\pi^*(t,s) = \frac{s\varphi_s(t,s)}{\varphi(t,s)}, \quad (t,s) \in \overline{\mathcal{N}} \ .$$
Then by (2.2.7), $\pi^* \in K$. Fix $(t^*, s^*) \in \mathcal{N}$ near (t_0, s_0) and set $S^*(u) := S_{t^*,s^*}(u)$,
$$\theta := \inf\{u \geq t^* : (u, S^*(u)) \notin \mathcal{N}\} \ .$$
Let
$$dZ^*(u) = Z^*(u)\sigma\pi^*(u, S^*(u))dW(u), \qquad u \in [t^*, \theta^*] \ ,$$
$$Z^*(t^*) = \varphi(t^*, s^*) \ .$$
By the Ito's rule, for $t^* < u < \theta^*$,
$$\begin{aligned} d[Z^*(u) - \varphi(u, S^*(u))] \ &= \ -\mathcal{L}\varphi(u, S^*(u)) \ du \\ &\quad +\sigma\pi^*(u, S^*(u))[Z^*(u) - \varphi(u, S^*(u))]dW(u) \ . \end{aligned}$$
Since $Z^*(t^*) - \varphi(t^*, S^*(t^*)) = 0$, and $-\mathcal{L}\varphi \geq 0$,
$$Z^*(u) \geq \varphi(u, S^*(u)) \ , \qquad \forall u \in [t^*, \theta] \ .$$
In particular,
$$Z^*(\theta) \geq \varphi(\theta, S^*(\theta) \ .$$
Also, $(\theta, S^*(\theta)) \notin \mathcal{N}$, and therefore
$$v^*(\theta, S^*(\theta)) \geq e^{-\delta} \ \varphi(\theta, S^*(\theta)) \ .$$
We combine all these to arrive at,
$$Z^*(\theta) \geq \varphi(\theta, S^*(\theta) \geq e^{\delta} \ v^*(\theta, S^*(\theta)) \geq e^{\delta} \ v(\theta, S^*(\theta)) \ .$$
Then,
$$Z^{\pi^*}_{t^*, e^{-\delta}\varphi(t^*,s^*)}(\theta) = e^{-\delta} Z^*(\theta) \geq v(\theta, S^*(\theta)) \ .$$
By the dynamic programming principle, (also since $\pi^*(\cdot) \in K$)),
$$e^{-\delta}\varphi(t^*, s^*) \geq v(t^*, s^*) \ .$$
Since the above inequality holds for every (t^*, s^*), by letting (t^*, s^*) tend to (t_0, s_0), we arrive at
$$e^{-\delta}\varphi(t_0, s_0) \geq v^*(t_0, s_0) \ .$$
But we assumed that $\varphi(t_0, s_0) = v^*(t_0, s_0)$.

Hence the inequality (2.2.6) must hold and v is a viscosity subsolution.
□

In the next subsection, we will study the terminal condition and then provide an explicit solution.

2.2.4 Terminal Condition or "Face-Lifting"

We showed that the value function $v(t,s)$ is a viscosity solution of

$$F(t,s,v(t,s),v_t(t,s),v_s(t,s),v_{ss}(t,s)) = 0, \qquad \forall s>0, \quad t<T \ .$$

In particular,

$$b\,v(t,s) - sv_s(t,s) \geq 0, \quad a\,v(t,s) - sv_s(t,s) \geq 0, \qquad \forall s>0, \quad t<T, \tag{2.2.8}$$

in the viscosity sense. Set

$$\overline{V}(s) := \limsup_{s'\to s, t'\uparrow T} v(t',s'), \qquad \underline{V}(s) := \liminf_{s'\to s, t'\uparrow T} v(t',s') \ .$$

Formally, since $v(t',.)$ satisfies (2.2.8) for every $t' < T$, we also expect $\overline{V}$ and $\underline{V}$ to satisfy (2.2.8) as well. However, given contingent claim G may not satisfy (2.2.8).

Example.
Consider a call option:

$$G(s) = (s-K)^+,$$

with $K = (-\infty, b)$ for some $b > 1$. Then, for $s > K$,

$$bG(s) - sG_s(s) = b(s-K)^+ - s \ .$$

This expression is negative for s near K. Note that the Black-Scholes replicating portfolio requires almost one share of the stock when time to maturity $(T-t)$ near zero and when $S(t) > K$ but close to K. Again at these points the price of the option is near zero. Hence to be able finance this replicating portfolio, the investor has to borrow an amount which is an arbitrarily large multiple of her wealth. So any borrowing constraints (i.e. any $b < +\infty$) makes the replicating portfolio inadmissible. □

We formally proceed and assume $\overline{V} = \underline{V} = V$. Formally, we expect V to satisfy (2.2.8) and also $V \geq 0$. Hence,

$$\begin{array}{rcl} bV(s) - sV_s(s) & \geq 0 & \geq -aV(s) - sV_s(s), \quad \forall s \geq 0, \\ V(s) & \geq & G(s), \quad \forall s \geq 0. \end{array} \tag{2.2.9}$$

Since we are looking for the *minimal* super replicating cost, it is natural to guess that $V(\cdot)$ is the minimal function satisfying (2.2.9). So we define

$$\hat{G}(s) := \inf\{H(s)\ :\ H \text{ satisfies (2.2.9) }\}\ . \qquad (2.2.10)$$

Example.

Here we compute $\hat{G}(s)$ corresponding to $G(s) = (s-K)^+$ and $K = (-\infty, b]$ for any $b > 1$. Then, $\hat{G}$ satisfies

$$\hat{G}(s)(s) \geq (K-s)^+\ ,$$
$$b\hat{G}(s)(s) - s\hat{G}_s(s) \geq 0\ . \qquad (2.2.11)$$

Assume $\hat{G}(\cdot)$ is smooth we integrate the second inequality to conclude

$$\hat{G}(s_0) \leq (\frac{s_0}{s_1})^b \hat{G}(s_1), \qquad \forall s_0 \geq s_1 > 0\ .$$

Again assuming that (2.2.11) holds on $[0, s^*]$ for some s^* we get

$$\hat{G}(s) = (\frac{s}{s^*})^b \hat{G}(s^*), \qquad \forall s \leq s^*\ .$$

Assume that $\hat{G}(s) = G(s)$ for $s \geq s^*$. Then, we have the following claim

$$\hat{G}(s) = h(s) := \begin{cases} (s-K)^+, & s \geq s^*, \\ (\frac{s}{s^*})^b (s^*-K)^+ & s < s^*, \end{cases}$$

where $s^* > K$ the unique point for which $h \in C^1$, i.e., $h_s(s^*) = G_s(s^*) = 1$. Then,

$$s^* = \frac{K}{b-1}$$

It is easy to show that h (with s^* as above) satisfies (2.2.9). It remains to show that h is the minimal function satisfying (2.2.9). This will be done through a general formula below. $\square$

Theorem 2.2.3 *Assume that G is nonnegative, continuous and grows at most linearly. Let $\hat{G}$ be given by* (2.2.10). *Then,*

$$\overline{V}(s) = \underline{V}(s) = \hat{G}(s)\ . \qquad (2.2.12)$$

In particular, $v(t,\cdot)$ converges to $\hat{G}$ uniformly on compact sets. Moreover,

$$\hat{G}(s) = \sup_{\nu \in \tilde{K}} e^{-\delta_K(\nu)} G(s, e^{-\nu}),$$

and

$$v(t,s) = E[\hat{G}(S_{t,s}(T))|\mathcal{F}_t],$$

i.e., v is the unconstrained (Black-Scholes) price of the modified claim $\hat{G}$.

This is proved in Broadie-Cvitanic-Soner [5]. A presentation is also given in Karatzas & Shreve [15] (Chapter 5.7). A proof based on the dual formulation is simpler and is given in both of the above references.

A PDE based proof of formulation $\hat{G}(\cdot)$ is also available. A complete proof in the case of a gamma constraint is given in the next Chapter. Here we only prove the final statement through a PDE argument. Set

$$u(t,s) = E[\hat{G}(S_{t,s}(T)) \mid \mathcal{F}_t] \ .$$

Then,

$$u_t = \frac{1}{2}\sigma^2 s^2 u_{ss} \qquad \forall t < T, s > 0,$$

and

$$u(T,s) = \hat{G}(s) \ .$$

It is clear that u is smooth. Set

$$w(t,s) := bu(t,s) - su_s(t,s) \ .$$

Then, since $u(T,s) = \hat{G}(s)$ satisfies the constraints,

$$w(T,s) \geq 0 \qquad \forall s \geq 0 \ .$$

Also using the equation satisfied by u, we calculate that

$$\begin{aligned} w_t &= bu_t - su_{st} = \frac{1}{2}\sigma^2 s^2 [bu_{ss}] - \frac{s}{2}\sigma^2 [s^2 u_{ss}]_s \\ &= \frac{1}{2}\sigma^2 s^2 [bu_{ss} - 2u_{ss} - s^2 u_{sss}] = \frac{1}{2}\sigma^2 s^2 w_{ss} \ . \end{aligned}$$

So by the Feynman-Kac formula,

$$bu(t,s) - su_s(t,s) = w(t,s) = E[w(t,s)|\mathcal{F}_t] \geq 0, \qquad \forall t \leq T, s \geq 0 \ .$$

Similarly, we can show that

$$au(t,s) + su_s(t,s) \geq 0, \qquad \forall s \leq T, s \geq 0 \ .$$

Therefore,

$$F(t,s,u(t,s),u_t(t,s),u_s(t,s),u_{ss}(t,s)) = 0, \qquad t < T, s > 0,$$

where F is as in the previous section. Since v, the value function, also solves this equation, and also $v(T,s) = u(T,s) = \hat{G}(s)$, by a comparison result for the above equation, we can conclude that $u = v$, the value function.

Chapter 3

Super-Replication with Gamma Constraints

This chapter is taken from Cheridito-Soner-Touzi [7, 8], and Soner-Touzi [19]. For the brevity of the presentation, again we consider a market with only one stock, and assume that the mean return rate $\mu = 0$, by a Girsanov transformation, and that the interest rate $r = 0$, by appropriately discounting all the prices. Then, the stock price $S(\cdot)$ follows

$$dS(t) = \sigma S(t) dW(t),$$

and the wealth process $Z(\cdot)$ solves

$$dZ(t) = Y(t) dS(t),$$

where the "portfolio process" $Y(\cdot)$ is a semi-martingale

$$dY(t) = dA(t) + \gamma(t) dS(t) \ .$$

The control processes $A(t) \in$ and $\gamma(\cdot)$ are required to satisfy

$$\Gamma_*(S(t)) \leq \gamma(t) \leq \Gamma^*(S(t)) \qquad \forall t, \ \text{a.s.} \ ,$$

for given functions Γ_* and Γ^*. The triplet $(Y(t) = y, A(\cdot), \gamma(\cdot)) = \nu$ is the control. Further restriction on $A(\cdot)$ will be replaced later. Then, the minimal super-replicating cost $v(t,s)$ is

$$v(t,s) = \inf\{z : \exists \nu \in \mathcal{A} \text{ s.t. } Z^{\nu}_{t,z} \geq G(S_{t,s}(T)) \text{ a.s. } \} \ .$$

3.1 Pure Upper Bound Case

We consider the case with no lower bound, i.e. $\Gamma_* = -\infty$. Since formally we expect $v_s(t, S(t)) = Y^*(t)$ to be the optimal portfolio, we also expect $\gamma^*(t) := v_{ss}(t, S(t))$. Hence, the gamma constraint translates into a differential inequality

$$v_{ss}(t,s) \leq \Gamma^*(s) \ .$$

In view of the portfolio constraint example, the expected DPE is

$$\min\{-v_t - \frac{1}{2}\sigma^2 s^2 v_{ss}; \quad -s^2 v_{ss} + s^2\Gamma^*(s)\} = 0 \ . \tag{3.1.1}$$

We could eliminate s^2 term in the second part, but we choose to write the equation this way for reasons that will be clear later.

Theorem 3.1.1 *Assume G is non-negative, lower semi-continuous and growing at most linearly. Then, v is a viscosity solution of* (3.1.1)

We will prove the sub and super solution parts separately.

3.1.1 Super solution

Consider a test function φ and $(t_0, s_0) \in (0,T) \times (0, \infty)$ so that

$$(v_* - \varphi)(t_0, s_0) = \min(v_* - \varphi) = 0 \ .$$

Choose $(t_n, s_n) \to (t_0, s_0)$ so that $v(t_n, s_n) \to v_*(t_0, s_0)$. Further choose $\nu_n = (y_n, A_n(\cdot), \gamma_n(\cdot))$ so that with $z_n := v(t_n, s_n) + 1/n^2$,

$$Z^{\nu_n}_{t_n,s_n}(t_n + \frac{1}{n}) \geq v(t_n + \frac{1}{n}, S_{t_n,s_n}(t_n + \frac{1}{n})) \quad \text{a.s.}$$

The existence of ν_n follows from the dynamic programming principle with stopping time $\theta = t_n + \frac{1}{n}$. Set $\theta := t_n + \frac{1}{n}$, $S_n(\cdot) = S_{t_n,s_n}(\cdot)$. Since $v \geq v_* \geq \varphi$, Ito's rule yield,

$$\begin{aligned} z_n + \int_{t_n}^{\theta_n} Y^{\nu_n}_{t_n,y_n}(u) dS(u) \ \ &\geq \ \ \varphi(t_n, s_n) \\ &\quad + \int_{t_n}^{\theta_n} [\mathcal{L}\varphi(u, S_n(u)) du + \varphi_s(u, S_n(u)) dS(u)] \ . \end{aligned} \tag{3.1.2}$$

Taking the expected value implies

$$\begin{aligned} n\int_{t_n}^{\theta_n} E(-\mathcal{L}\varphi(u,S_n(u)))du &\geq n[\varphi(t_n,s_n)-z_n] \\ &= n[\varphi - v](t_n,s_n) - \frac{1}{n} \ . \end{aligned}$$

We could choose (t_n, s_n) so that $n[\varphi - v](t_n, s_n) \to 0$. Hence, by taking the limit we obtain,

$$-\mathcal{L}\varphi(t_0,s_0) \geq 0$$

To obtain the second inequality, we return to (3.1.2) and use the Ito's rule on φ_s and the dynamics of $Y(\cdot)$. The result is

$$\begin{aligned} \alpha_n &+ \int_{t_n}^{\theta_n} a_n(u)du + \int_{t_n}^{\theta_n}\int_{u}^{\theta_n} b_n(t)dtdS(u) \\ &\quad + \int_{t_n}^{\theta_n}\int_{u}^{\theta_n} c_n(t)dS(t)dS(u) + \int_{t_n}^{\theta_n} d_n dS(u) \geq 0 \ , \end{aligned}$$

where

$$\begin{aligned} \alpha_n &= z_n - \varphi(t_n,s_n), \\ a_n(u) &= -\mathcal{L}\varphi(u,S_n(u)), \\ b_n(u) &= dA_n(u) - \mathcal{L}\varphi_s(u,S_n(u)), \\ c_n(u) &= \gamma_n(u) - \varphi_{ss}(u,S_n(u)), \\ d_n &= y_n - \varphi_s(t_n,s_n) \ . \end{aligned}$$

We now need to define the set of admissible controls in a precise way and then use our results on double stochastic integrals. We refer to the paper by Cheridito-Soner-Touzi for the definition of the set of admissible controls. Then, we can use the results of the next subsection to conclude that

$$-\mathcal{L}\varphi(t_0,s_0) = \limsup_{u\downarrow t} a_n(u) \geq 0 \ ,$$

and

$$\limsup_{u\downarrow t} c_n(u) \geq 0 \ .$$

In particular, this implies that

$$-\varphi_{ss}(t_0,s_0) \leq \Gamma^*(s_0) \ .$$

Hence, v is a viscosity super solution of

$$\min\{-v_t - \frac{1}{2}\sigma^2 s^2 v_{ss}; \quad -s^2 v_{ss} + s^2\Gamma^*(s)\} \geq 0 \ .$$

3.1.2 Subsolution

Consider a smooth φ and (t_0, s_0) so that

$$(v^* - \varphi)(t_0, s_0) = \max(v^* - \varphi) = 0$$

Suppose that

$$\min\{-\varphi_t(t_0, s_0) - \frac{1}{2}\sigma^2 s_0^2 \varphi_{ss}(t_0, s_0) \ ; \ -s_0^2\varphi_{ss}(t_0, s_0) + s_0^2\Gamma^*(s_0)\} > 0 \ .$$

We will obtain a contradiction to prove the subsolution property. We proceed exactly as in the constrained portfolio case using the controls

$$y = \varphi_s(t_0, s_0), \quad dA = \mathcal{L}\varphi_s(u, S(u))du, \quad \gamma = \varphi_{ss}(u, S(u)) \ .$$

Then, in a neighborhood of (t_0, s_0), $Y(u) = \varphi_s(u, S(u))$ and $\gamma(\cdot)$ satisfies the constraint. Since $-\mathcal{L}\varphi > 0$ in this neighborhood, we can proceed exactly as in the constrained portfolio problem.

3.1.3 Terminal Condition

In this subsection, we will show that the terminal condition is satisfied with a modified function $\hat{G}$. This is a similar result as in the portfolio constraint case and the chief reason for modifying the terminal data is to make it compatible with the constraints. Assume

$$0 \leq s^2\Gamma^*(s) \leq c^*(s^2 + 1) \ . \tag{3.1.3}$$

Theorem 3.1.2 *Under the previous assumptions on G,*

$$\lim_{t' \to T, s' \to s} v(t', s') = \hat{G}(s),$$

where $\hat{G}$ is smallest function satisfying (i) $\hat{G} \geq G$ (ii) $\hat{G}_{ss}(s) \leq \Gamma^*(s)$. Let $\gamma^*(s)$ be a smooth function satisfying

$$\gamma^*_{ss} = \Gamma^*(s) \ .$$

Then,

$$\hat{G}(s) = h_{conc}(s) + \gamma^*(s) \ ,$$
$$h(s) := G(s) - \gamma^*(s) \ ,$$

and $h_{conc}(s)$ is the concave envelope of h, i.e., the smallest concave function above h.

Proof: Consider a sequence $(t_n, s_n) \to (T, s_0)$. Choose $\nu^n \in \mathcal{A}(t_n)$ so that

$$Z^{\nu_n}_{t_n, v(t_n,s_n)+\frac{1}{n}}(T) \geq G(S_{t_n,s_n}(T)) \ .$$

Take the expected value to arrive at

$$v(t_n, s_n) + \frac{1}{n} \geq E(G_{t_n,s_n}(T) \mid \mathcal{F}_{t_n}) \ .$$

Hence, by the Fatou's lemma,

$$\underline{V}(s_0) := \liminf_{(t_n,s_n)\to(T,s_0)} v(t_n, s_n) \geq G(s_0) \ .$$

Moreover, since v is a super solution of (3.1.1), for every t

$$v_{ss}(t, s) \leq \Gamma^*(s) \quad \text{in the viscosity sense} \ .$$

By the stability of the viscosity property,

$$\underline{V}_{ss}(s) \leq \Gamma^*(s) \quad \text{in the viscosity sense} \ .$$

Hence, we proved that $\underline{V}$ is a viscosity supersolution of

$$\min\{\underline{V}(s) - G(s); \quad -V_{ss}(s) + \Gamma^*(s)\} \geq 0 \ .$$

Set $w(s) = \underline{V}(s) - \gamma^*(s)$. Then, w is a viscosity super solution of

$$\min\{w(s) - h(s); \quad -w_{ss}(s)\} \geq 0 \ .$$

Therefore, w is concave and $w \geq h$. Hence, $w \geq h_{conc}$ and consequently

$$\underline{V}(s) = w(s) + \gamma^*(s) \geq h_{conc}(s) + \gamma^*(s) = \hat{G}(s) \ .$$

To prove the reverse inequality fix (t,s). For $\lambda > 0$ set

$$\gamma^\lambda(t,s) := (1-\lambda)\gamma^*(s) + c(T-t)s^2/2 \ .$$

Set

$$\gamma(u) = \gamma^\lambda_{ss}(u, S(u)), \quad dA(u) = \mathcal{L}\gamma^\lambda_s(u, S(u)), \quad y_0 = \gamma^\lambda_s(t,s) + p_0,$$

where p_0 is any point in the subdifferential of $h_{\lambda,conc}$, $h_\lambda = G - (1-\lambda)\gamma^*$. Then,

$$p_0(s'-s) + h_{\lambda,conc}(s) \geq h_{\lambda,conc}(s'), \qquad \forall s' \ .$$

For any z, consider $Z^\nu_{t,z}$ with $\nu = (y_0, A(\cdot), \gamma(\cdot))$. Note for t near T, $\nu \in \mathcal{A}(t)$,

$$\begin{aligned}
Z^\nu_{t,z}(T) &= z + \int_t^T (\gamma^\lambda_s(t,s) + p_0) dS(u) \\
&\quad + \int_t^T [\int_t^u \mathcal{L}\gamma^\lambda_s(r, S(r))dr + \gamma^\lambda_{ss}(r, S(r))dS(r)]dS \\
&= z + p_0 \ (S(T) - s) + \int_t^T \gamma^\lambda_s(u, S(u))dS(u) \\
&\geq z + h_{\lambda,conc}(S(T)) - h_{\lambda,conc}(s) \\
&\quad + \gamma^\lambda(S(T), T) - \gamma^\lambda(t,s) - \int_t^T \mathcal{L}\gamma^\lambda(u, S(u))du \ .
\end{aligned}$$

We directly calculate that with c^* as in (3.1.3),

$$\begin{aligned}
\mathcal{L}\gamma^\lambda(t,s) &= -cs^2 + \frac{1}{2}\sigma^2 s^2((1-\lambda)\gamma^*_{ss} + c(T-t)) \\
&= -s^2\{c - \frac{1}{2}\sigma^2[(1-\lambda)\Gamma^*(s) + +c(T-t)]\} \\
&\leq -s^2\{c - \frac{1}{2}\sigma^2[(1-\lambda)c^* + c(T-t)]\} + \frac{1}{2}\sigma^2 c^* \\
&\leq \frac{1}{2}\sigma^2 c^* \ ,
\end{aligned}$$

provided that t is sufficiently near T and c is sufficiently large. Hence, with

$$z_0 = h_{\lambda,conc}(s) + \gamma^\lambda(t,s) + \frac{1}{2}\sigma^2 c^*(T-t),$$

we have

$$\begin{aligned} Z^{\nu}_{t,z_0} &\geq h_{\lambda,conc}(S(T)) + \gamma^{\lambda}(S(T),T) \\ &\geq G(S(T)) - (1-\lambda)\gamma^*(S(T)) + (1-\lambda)\gamma^*(S(T)) \\ &= G(S(T)) \ . \end{aligned}$$

Therefore,

$$z_0 = h_{\lambda,conc}(s) + \gamma^{\lambda}(t,s) + \frac{1}{2}\sigma^2 c^*(T-t) \geq v(t,s) \ ,$$

for all $\lambda > 0$, c large and t sufficiently close to T. Hence,

$$\begin{aligned} \limsup_{(t,s)\to(T,s_0)} v(t,s) &\leq h_{\lambda,conc}(s_0) + \gamma^{\lambda}(T,s_0) \qquad \forall\lambda > 0 \\ &= h_{\lambda,conc}(s_0) + (1-\lambda)\gamma^*(s_0) \\ &= (G - (1-\lambda)\gamma^*)_{conc}(s_0) + (1-\lambda)\gamma^*(s_0) \\ &= \hat{G}_{\lambda}(s_0) \ . \end{aligned}$$

It is easy to prove that

$$\lim_{\lambda\downarrow 0} \hat{G}_{\lambda}(s_0) = \hat{G}(s_0) \ .$$

□

3.2 Double Stochastic Integrals

In this section, we study the asymptotic behavior of certain stochastic integrals, as $t \downarrow 0$. These properties were used in the derivation of the viscosity property of the value function. Here we only give some of the results and the proofs. A detailed discussion is given in the recent manuscript of Cheridito, Soner and Touzi.

For a predictable, bounded process b, let

$$V^b(t) := \int_0^t \int_0^u b(\rho)dW(\rho)dW(u) \ ,$$

$$h(t) := t \ln\ln 1/t \ .$$

For another predictable, bounded process m, let

$$M(t) := \int_0^t m(u)dW(u),$$

$$V_m^b := \int_0^t \int_0^u b(\rho) dM(\rho) dM(u) \ .$$

In the easy case when $b(t) = \beta$, $t \geq 0$, for some constant $\beta \in \mathbb{R}$, we have

$$V^b(t) = \frac{\beta}{2} \left(W^2(t) - t \right) \ , t \geq 0 \, .$$

If $\beta \geq 0$, it follows from the law of the iterated logarithm for Brownian motion that,

$$\limsup_{t \searrow 0} \frac{2V^\beta(t)}{h(t)} = \beta \, , \tag{3.2.4}$$

where

$$h(t) := 2t \log \log \frac{1}{t} \, , \, t > 0 \, ,$$

and the equality in (3.2.4) is understood in the almost sure sense. On the other hand, if $\beta < 0$, it can be deduced from the fact that almost all paths of a one-dimensional standard Brownian motion cross zero on all time intervals $(0, \varepsilon]$, $\varepsilon > 0$, that

$$\limsup_{t \searrow 0} \frac{2V^\beta(t)}{t} = -\beta \, . \tag{3.2.5}$$

The purpose of this section is to derive formulae similar to (3.2.4) and (3.2.5) when $b = \{b(t) \, , \, t \geq 0\}$ a predictable matrix-valued stochastic process.

Lemma 3.2.1 *Let λ and T be two positive parameters with $2\lambda T < 1$ and $\{b(t) \, , \, t \geq 0\}$ an $\mathcal{M}^d$-valued, $\mathbb{F}$-predictable process such that $|b(t)| \leq 1$, for all $t \geq 0$. Then*

$$\mathrm{E} \left[\exp \left(2\lambda V^b(T) \right) \right] \leq \mathrm{E} \left[\exp \left(2\lambda V^{I_d}(T) \right) \right] \ .$$

Proof. We prove this lemma with a standard argument from the theory of stochastic control. We define the processes

$$Y^b(r) := Y(0) + \int_0^r b(u) dW(u) \ \text{ and } \ Z^b(t) := Z(0) + \int_0^t (Y^b(r))^T dW(r) \, , \quad t \geq 0 \, ,$$

where $Y(0) \in \Re^d$ and $Z(0) \in \Re$ are some given initial data. Observe that $V^b(t) = Z^b(t)$ when $Y(0) = 0$ and $Z(0) = 0$. We split the argument into three steps.

Step 1: It can easily be checked that

$$\mathrm{E}\left[\exp\left(2\lambda Z^{I_d}(T)\right)\big|\,\mathcal{F}_t\right] = f\left(t, Y^{I_d}(t), Z^{I_d}(t)\right)\ , \qquad (3.2.6)$$

where, for $t \in [0,T]$, $y \in \Re^d$ and $z \in \Re$, the function f is given by

$$\begin{aligned} f(t,y,z) \ &:=\ \mathrm{E}\left[\exp\left(2\lambda\left\{z + \int_t^T (y + W(r) - W(t))^T\, dW(r)\right\}\right)\right] \\ &=\ \exp(2\lambda z)\,\mathrm{E}\left[\exp\left(\lambda\{2y^T W(T-t) + |W(T-t)|^2 - d(T-t)\}\right)\right] \\ &=\ \mu^{d/2}\exp\left[2\lambda z - d\lambda(T-t) + 2\mu\lambda^2(T-t)|y|^2\right]\ , \end{aligned}$$

and $\mu := [1 - 2\lambda(T-t)]^{-1}$. Observe that the function f is strictly convex in y and

$$D^2_{yz} f(t,y,z) \ :=\ \frac{\partial^2 f}{\partial y \partial z}(t,y,z) \ =\ g^2(t,y,z)\, y \qquad (3.2.7)$$

where $g^2 := 8\mu\lambda^3(T-t)\, f$ is a positive function of (t,y,z).

Step 2: For a matrix $\beta \in \mathcal{M}^d$, we denote by $\mathcal{L}^\beta$ the Dynkin operator associated to the process $\left(Y^b, Z^b\right)$, that is,

$$\mathcal{L}^\beta \ :=\ D_t + \frac{1}{2} tr\left[\beta\beta^T D^2_{yy}\right] + \frac{1}{2}|y|^2 D^2_{zz} + (\beta y)^T D^2_{yz}\,,$$

where $D_{\cdot}$ and $D^2_{\cdot\cdot}$ denote the gradient and the Hessian operators with respect to the indexed variables. In this step, we intend to prove that for all $t \in [0,T]$, $y \in \Re^d$ and $z \in \Re$,

$$\max_{\beta\in\mathcal{M}^d,\ |\beta|\le 1} \mathcal{L}^\beta f(t,y,z) \ =\ \mathcal{L}^{I_d} f(t,y,z) \ =\ 0\,. \qquad (3.2.8)$$

The second equality can be derived from the fact that the process

$$f\left(t, Y^{I_d}(t), Z^{I_d}(t)\right)\ ,\ t \in [0,T]\,,$$

is a martingale, which can easily be seen from (3.2.6). The first equality follows from the following two observations: First, note that for each $\beta \in \mathcal{M}^d$ such that $|\beta| \le 1$, the matrix $I_d - \beta\beta^T$ is in $\mathcal{S}^d_+$. Therefore, there exists a $\gamma \in \mathcal{S}^d_+$ such that

$$I_d - \beta\beta^T = \gamma^2\,.$$

Since f is convex in y, the Hessian matrix $D^2_{yy}f$ is also in $\mathcal{S}^d_+$. It follows that $\gamma D^2_{yy}f(t,x,y)\gamma \in \mathcal{S}^d_+$, and therefore,

$$\begin{aligned} tr[D^2_{yy}f(t,x,y)] &- tr[\beta\beta^T D^2_{yy}f(t,x,y)] = tr[(I_d - \beta\beta^T)D^2_{yy}f(t,x,y)] \\ &= tr[\gamma D^2_{yy}f(t,x,y)\gamma] \geq 0\,. \end{aligned} \tag{3.2.9}$$

Secondly, it follows from (3.2.7) and the Cauchy-Schwartz inequality that, for all $\beta \in \mathcal{M}^d$ such that $|\beta| \leq 1$,

$$\begin{aligned} (\beta y)^T D^2_{yz}f(t,y,z) &= g^2(t,y,z)(\beta y)^T y \leq g^2(t,y,z)|y|^2 \\ &= y^T D^2_{yz}f(t,y,z)\,. \end{aligned} \tag{3.2.10}$$

Together, (3.2.9) and (3.2.10) imply the first equality in (3.2.8).

Step 3: Let $\{b(t)\,,\, t \geq 0\}$ be an $\mathcal{M}^d$-valued, $\mathbb{F}$-predictable process such that $|b(t)| \leq 1$ for all $t \geq 0$. We define the sequence of stopping times

$$\tau_k \;:=\; T \wedge \inf\left\{t \geq 0 \;:\; |Y^b(t)| + |Z^b(t)| \geq k\right\}\;,\quad k \in \mathbb{N}\;.$$

It follows from Itô's lemma and (3.2.8) that

$$\begin{aligned} f(0,Y(0),Z(0)) &= f\left(\tau_k, Y^b(\tau_k), Z^b(\tau_k)\right) - \int_0^{\tau_k} \mathcal{L}^{b(t)} f\left(t, Y^b(t), Z^b(t)\right) dt \\ &\quad - \int_0^{\tau_k} [(D_y f)^T b + (D_z f) y^T]\left(t, Y^b(t), Z^b(t)\right) dW(t) \\ &\geq f\left(\tau_k, Y^b(\tau_k), Z^b(\tau_k)\right) \\ &\quad - \int_0^{\tau_k} [(D_y f)^T b + (D_z f) y^T]\left(t, Y^b(t), Z^b(t)\right) dW(t)\;. \end{aligned}$$

Taking expected values and sending k to infinity, we get by Fatou's lemma,

$$\begin{aligned} &\mathrm{E}\left[\exp\left(2\lambda Z^{I_d}(T)\right)\right] = f(0,Y(0),Z(0)) \\ \geq\;& \liminf_{k\to\infty}\; \mathrm{E}\left[f\left(\tau_k, Y^b(\tau_k), Z^b(\tau_k)\right)\right] \\ \geq\;& \mathrm{E}\left[f\left(T, Y^b(T), Z^b(T)\right)\right] \;=\; \mathrm{E}\left[\exp\left(2\lambda Z^b(T)\right)\right]\;, \end{aligned}$$

which proves the lemma. □

Theorem 3.2.2
a) *Let* $\{b(t)\,,\, t \geq 0\}$ *be an* $\mathcal{M}^d$*-valued,* $\mathbb{F}$*-predictable process such that* $|b(t)| \leq 1$ *for all* $t \geq 0$*. Then*

$$\limsup_{t \searrow 0} \frac{|2V^b(t)|}{h(t)} \leq 1\,.$$

b) *Let* $\beta \in \mathcal{S}^d$ *with largest eigenvalue* $\lambda^*(\beta) \geq 0$*. If* $\{b(t)\,,\, t \geq 0\}$ *is a bounded,* $\mathcal{S}^d$*-valued,* $\mathbb{F}$*-predictable process such that* $b(t) \geq \beta$ *for all* $t \geq 0$*, then*

$$\limsup_{t \searrow 0} \frac{2V^b(t)}{h(t)} \geq \lambda^*(\beta)\,.$$

Proof.
a) Let $T > 0$ and $\lambda > 0$ be such that $2\lambda T < 1$. It follows from Doob's maximal inequality for submartingales and Lemma 3.2.1 that for all $\alpha \geq 0$,

$$\begin{aligned} P\left[\sup_{0 \leq t \leq T} 2V^b(t) \;\geq\; \alpha\right] &= P\left[\sup_{0 \leq t \leq T} \exp(2\lambda V^b(t)) \;\geq\; \exp(\lambda\alpha)\right] \\ &\leq \exp\left(-\lambda\alpha\right) \mathrm{E}\left[\exp\left(2\lambda V^b(T)\right)\right] \\ &\leq \exp\left(-\lambda\alpha\right) \mathrm{E}\left[\exp\left(2\lambda V^{I_d}(T)\right)\right] \\ &= \exp\left(-\lambda\alpha\right) \exp\left(-\lambda d T\right) \left(1 - 2\lambda T\right)^{-\frac{d}{2}}\,. \end{aligned} \tag{3.2.11}$$

Now, take θ, $\eta \in (0,1)$, and set for all $k \in \mathbb{N}$,

$$\alpha_k \;:=\; (1+\eta)^2 h(\theta^k) \quad \text{and} \quad \lambda_k \;:=\; [2\theta^k(1+\eta)]^{-1}\,.$$

It follows from (3.2.11) that for all $k \in \mathbb{N}$,

$$P\left[\sup_{0 \leq t \leq \theta^k} 2V^b(t) \;\geq\; (1+\eta)^2 h(\theta^k)\right] \;\leq\; \exp\left(-\frac{d}{2(1+\eta)}\right)\left(1 + \frac{1}{\eta}\right)^{d/2} (k \log \frac{1}{\theta})^{-(1+\eta)}\,.$$

Since

$$\sum_{k=1}^{\infty} k^{-(1+\eta)} < \infty\,,$$

it follows from the Borel-Cantelli lemma that, for almost all $\omega \in \Omega$, there exists a natural number $K^{\theta,\eta}(\omega)$ such that for all $k \geq K^{\theta,\eta}(\omega)$,

$$\sup_{0 \leq t \leq \theta^k} 2V^b(t,\omega) \;<\; (1+\eta)^2 h(\theta^k)\,.$$

In particular, for all $t \in (\theta^{k+1}, \theta^k]$,

$$2V^b(t,\omega) \;<\; (1+\eta)^2 h(\theta^k) \;\leq\; (1+\eta)^2 \frac{h(t)}{\theta}\,.$$

Hence,

$$\limsup_{t \searrow 0} \frac{2V^b(t)}{h(t)} \leq \frac{(1+\eta)^2}{\theta}\,.$$

By letting θ tend to one and η to zero along the rationals, we conclude that

$$\limsup_{t \searrow 0} \frac{2V^b(t)}{h(t)} \leq 1\,.$$

On the other hand,

$$\liminf_{t \searrow 0} \frac{2V^b(t)}{h(t)} \;=\; -\limsup_{t \searrow 0} \frac{2V^{-b}(t)}{h(t)} \geq -1\,,$$

and the proof of part a) is complete.
b) There exists a constant $c > 0$ such that for all $t \geq 0$,

$$cI_d \geq b(t) \geq \beta \geq -cI_d\,, \tag{3.2.12}$$

and there exists an orthogonal $d \times d$-matrix U such that

$$\tilde{\beta} := U\beta U^T = \operatorname{diag}[\lambda^*(\beta), \lambda_2, \ldots, \lambda_d]\,,$$

where $\lambda^*(\beta) \geq \lambda_2 \geq \cdots \geq \lambda_d$ are the ordered eigenvalues of the matrix β. Let

$$\tilde{\gamma} := \operatorname{diag}[3c, c, \ldots, c] \quad \text{and} \quad \gamma := U^T \tilde{\gamma} U\,.$$

It follows from (3.2.12) that for all $t \geq 0$,

$$\gamma - \beta \geq \gamma - b(t) \geq 0\,,$$

which implies that

$$|\gamma - b(t)| \leq |\gamma - \beta| = \lambda^*(\gamma - \beta) = \lambda^*(\tilde{\gamma} - \tilde{\beta}) = 3c - \lambda^*(\beta)\,.$$

Hence, by part a),

$$\limsup_{t \searrow 0} \frac{2V^{\gamma - b}(t)}{h(t)} \leq 3c - \lambda^*(\beta)\,. \tag{3.2.13}$$

Note that the transformed Brownian motion,

$$\tilde{W}(t) := UW(t)\,,\ t \geq 0\,,$$

is again a d-dimensional standard Brownian motion and

$$\begin{aligned}\limsup_{t\searrow 0} \frac{2V^{\gamma}(t)}{h(t)} &= \limsup_{t\searrow 0} \frac{W(t)^T\gamma W(t) - tr(\gamma)t}{h(t)} \\ &= \limsup_{t\searrow 0} \frac{\tilde{W}(t)^T\tilde{\gamma}\tilde{W}(t) - tr(\gamma)t}{h(t)} \\ &= \limsup_{t\searrow 0} \frac{\tilde{W}(t)^T\tilde{\gamma}\tilde{W}(t)}{h(t)} \\ &\geq \limsup_{t\searrow 0} 3c\frac{(\tilde{W}_1(t))^2}{h(t)} = 3c\,.\end{aligned} \tag{3.2.14}$$

It follows from (3.2.14) and (3.2.13) that

$$\limsup_{t\searrow 0} \frac{2V^{b}(t)}{h(t)} \geq \limsup_{t\searrow 0} \frac{2V^{\gamma}(t)}{h(t)} - \limsup_{t\searrow 0} \frac{2V^{\gamma-b}(t)}{h(t)} \geq 3c-(3c-\lambda^*(\beta)) = \lambda^*(\beta)\,,$$

which proves part b) of the theorem.

□

Using the above results one can prove the following lower bound for V^b_m.

Theorem 3.2.3 *Assume that*

$$\sup_u \frac{E|m(u) - m(0)|^4}{u^4} \leq \infty\ .$$

Then,

$$\limsup_{t\downarrow 0} \frac{V^b_m(t)}{h(t)} \geq a\ ,$$

for every $b \in L^\infty$ *satisfying* $m(0)b(0)m(u) \geq a > 0$.

Theorem 3.2.4 *Suppose that* $c \in L^\infty$ *and* $E|m(u)|^4 \leq c^*$. *Then, for all* $\varepsilon > 0$,

$$\limsup_{t\downarrow 0} \frac{|\int_0^t(\int_0^u a(\rho)d\rho)dM(u)|}{t^{3/2+\varepsilon}} = 0\ .$$

Proofs of these results can be found in Cheridito-Soner-Touzi.

Lemma 3.2.5 *Suppose that $b \in L^\infty$ and for some $\delta > 0$*

$$\sup_{u \geq 0} \frac{E|b(u) - b(0)|^2}{u^{2\delta}} < \infty \ .$$

Then,

$$\liminf_{t \downarrow 0} \frac{V^b(t)}{t} = -\frac{1}{2} b(0) \ .$$

Proof: Set $\tilde{b}(u) := b(u) - b(0)$. Then, $E|\tilde{b}(u)|^2/u^{2\delta} < \infty$ and

$$V^b(t) = \frac{1}{2} b(0) W^2(t) - \frac{1}{2} b(0) t + V^{\tilde{b}}(t) \ .$$

It can be shown that $|V^{\tilde{b}}(t)| = 0(t)$. Hence,

$$\liminf_{t \downarrow 0} \frac{V^b(t)}{t} = -\frac{1}{2} b(0) + \liminf_{t \downarrow 0} \frac{1}{2} b(0) \frac{W^2(t)}{t} = -\frac{1}{2} b(0) \ .$$

□

3.3 General Gamma Constraint

Here we consider the constraint

$$\Gamma_*(S(u)) \leq \gamma(u) \leq \Gamma^*(S(u)) \ .$$

Let

$$H(v_t, v_s, v_{ss}, s) = \min\{-v_t - \frac{1}{2}\sigma^2 s^2 v_{ss};\ -v_{ss} + \Gamma^*(s),\ v_{ss} - \Gamma_*(s)\} \ ,$$

and $\hat{H}$ be the parabolic majorant of H, i.e.,

$$\hat{H}(v_t, v_s, v_{ss}, s) := \sup\{H(v_t, v_s, v_{ss} + Q, s) : \quad Q \geq 0\} \ .$$

Theorem 3.3.1 *v is a viscosity solution of*

$$\hat{H}(v_t, v_s, v_{ss}, s) = 0, \qquad (t, s) \in (0, T) \times (0, \infty)$$

The super solution property is already proved. The subsolution property is proved almost exactly as in the upper gamma constraint case; cf [Cheridito-Soner-Touzi].

Terminal condition is the same in the pure upper bound case. As before

$$\hat{G}(s) = (G - \gamma^*)_{conc}(s) + \gamma^*(s) \ ,$$

where γ^* satisfies, $\gamma^*_{ss}(s) = \Gamma^*(s)$. Note that the lower gamma bound does not effect $\hat{G}$.

Proposition 3.3.2

$$\lim_{(t',s')\to(T,s)} v(t',s') = \hat{G}(s)$$

Proof is exactly as in the upper gamma bound case.

Theorem 3.3.3 *Assume* $0 \le G(s) \le c^*[1+s]$. *Then,* v *is the unique viscosity solution of* $\hat{H} = 0$ *together with the boundary condition* $v(T,s) = \hat{G}(s)$.

Proof is given in Cheridito-Soner-Touzi.

Example. Consider the problem with $\Gamma_*(s) \equiv 0, \Gamma^*(s) \equiv +\infty, G(s) = s \wedge 1$. For $s \le 1, z = s, y_0 = 1, A \equiv 0, \gamma \equiv 0$ yield,

$$Y(u) \equiv 1, \quad Z^{\nu}_{t,s}(u) = s + \int_t^u dS(\rho) = S_{t,s}(u) \ .$$

Hence, $Z^{\nu}_{t,s}(T) = S_{t,s}(T) \ge S_{t,s}(T) \wedge 1$ and for $s < 1, v(t,s) \le s$. For $s \ge 1$, let $z = 1, y_0 = 0, A \equiv 0, \gamma \equiv 0$. Then,

$$Y(u) \equiv 0, \quad Z^{\nu}_{t,1}(u) = 1 \ ,$$

and $Z^{\nu}_{t,1}(T) = 1 \ge S_{t,s}(T) \wedge 1$. So for $s \ge 1$ $v(t,s) \le 1$.

Hence, $v(t,s) \le G(s)$. We can check that

$$\hat{H}(G_t, G_s, G_{ss}) = \sup_{Q \ge 0} \{ -\frac{1}{2}\sigma^2 s^2 (G_{ss} + Q);\ (G_{ss} + Q)\} = 0 \ .$$

By the uniqueness, we conclude that $v = G$, and the buy and hold strategy described earlier is optimal. □

3.4 Examples

In some cases it is possible to compute the solution. In this section, we give several examples to illustrate the theory.

3.4.1 European Call Option

$$G(s) = (s - K)^+ \ .$$

(a). Let us first consider the portfolio constraint:

$$-Lv \leq sv_s \leq Uv \ .$$

where $(U - 1)$ is the fraction of our wealth we are allowed to borrow (i.e., shortsell the money market) and L is the shortsell constraint. According to our results,

$$v(t, s) = E[\hat{G}(S_{t,s}(T))] \ ,$$

where $\hat{G}$ is the smallest function satisfying $i)\hat{G} \geq G$, $ii) - L\hat{G} \leq s\hat{G}_s \leq U\hat{G}$. By observation, we see that only the upper bound is relevant and that there exists $s^* > K$ so that

$$\hat{G}(s) = G(s) = (s - K) \qquad \forall s \geq s^*$$

Moreover, for $s \leq s^*$ the constraint $s\hat{G}(s) = U\hat{G}(s)$ saturates, i.e.

$$s\hat{G}(s) = U\hat{G}(s) \qquad \forall s \geq s^*$$

Hence

$$\hat{G}(s) = (\frac{s}{s^*})^U \hat{G}(s^*) = (s^* - K)(\frac{s}{s^*})^U$$

Now we compute s^*, by imposing that $\hat{G} \in C^1$. So

$$\hat{G}_s(s^*) = G_s(s^*) = 1 \Rightarrow U(s^* - K)(\frac{1}{s^*}) = 1 \ .$$

The result is

$$s^* = \frac{U}{(v - 1)} K \ .$$

Therefore,

$$\hat{G}(s) = \begin{cases} (s - K), & \text{if} \quad s \geq s^* \\ (\frac{U-1}{K})^{U-1} U^{-U} s^U, & \text{if} \quad 0 \leq s \leq s^* \ . \end{cases}$$

Set,

$$G_{premium}(s) := \hat{G}(s) - G(s) \ .$$

Then,

$$v(t,s) = v_{BS}(t,s) + v_{pr}(t,s) \ ,$$

where $v_{BS}(t,s) := EG(t,s)$ is the Black Scholes price, and

$$v_{pr}(t,s) = E[\hat{G}(s) - G(S(T))]$$

Moreover, v_{pr} can be explicitly calculated (and its derivative) in terms of the error function, as in the Black-Scholes Case.
(b). Now consider the Gamma constraint

$$-\gamma_* v \leq s^2 v_{ss} \leq \gamma^* v \ .$$

Again, since G is convex, the lower bound is irrelevant and the modified $\hat{G}$ is given as in part (a) with

$$U(U-1) = \gamma^* \Rightarrow U = \frac{1+\sqrt{1+4\gamma^*}}{2} \ .$$

So, it is interesting to note that the minimal super-replication cost of both constraints agree provided that U and γ^* are related as above.

3.4.2 European Put Option:

$$G(s) = (K-s)^+ \ .$$

(a). Again let us consider the portfolio constraint first. In this case, since G is decreasing, only lower bound is active. We compute as in the Call Option Case:

$$\hat{G}(s) = \begin{cases} (K-s), & \text{if} \quad 0 \leq s \leq s^* \ , \\ L^L(\frac{K}{L+1})^{L+1} s^{-L}, & \text{if} \quad s \geq s^* \ , \end{cases}$$

where

$$s_* = \frac{L}{L+1} K \ .$$

(b). Consider the gamma constraint

$$-\gamma_* v \leq s^2 v_{ss} \leq \gamma^* v \ .$$

Again lower bound inactive, and the solution is as in the portfolio case with

$$L = \gamma^*(\gamma^* + 1) \ .$$

3.4.3 European Digital Option

$$G(s) = \begin{cases} 1, & s \geq K \ , \\ 0, & s \leq K \ . \end{cases}$$

(a). Again, we first consider the portfolio constraint. It can be verified that

$$\hat{G}(s) = \begin{cases} (\frac{s}{K})^U, & 0 \leq s \leq K \ , \\ 1, & s \geq K \ . \end{cases}$$

If we split the price as before:

$$v(t,s) = v_{BS}(t,s) + v_{pr}(t,s) \ .$$

We have

$$v_{BS}(t,s) = \mathbb{P}(S_{t,s}(T) \geq 1) = N(d_2) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-x^2/2} dx \ ,$$

$$d_2 = \frac{\ln(\frac{s}{k}) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \ ,$$

$$\begin{aligned} v_{pr}(t,s) &= \frac{1}{\sqrt{2\pi}} \int^{-d_2} (\frac{s}{K})^U e^{U[\sigma\sqrt{T-t}x - \frac{1}{2}\sigma^2(T-t)]} e^{-\frac{1}{2}x^2} dx \\ &= (\frac{s}{K})^U e^{\frac{u(u-1)}{2}\sigma^2(T-t)} [\frac{1}{2} - N(d_2)] \ . \end{aligned}$$

We can also compute the corresponding delta's.
(b). Once again, the gamma constraint

$$s^2 v_{ss} \leq \gamma^* v$$

yields the same $\hat{G}$ as in the portfolio constraint with

$$\gamma^* = U(U-1) \Rightarrow U = \frac{1 + \sqrt{1 + 4\gamma^*}}{2} \ .$$

If there is *no lower gamma bound*, $v = v_{BS} + v_{pr}$.
(c). Suppose now that there is *no upper bound*,

$$s^2 v_{ss} \geq -\gamma_* v \ .$$

Then the dynamic programming equation is

$$\sup_{Q\geq 0}\min\{-v_t-\frac{\sigma^2}{2}(s^2v_{ss}+Q);\ (s^2v_{ss}+Q)+\gamma_*v\}=0\quad s>0,t<T\ .$$

$$v(T,s)=G(s)\ .$$

By calculus, we see that this is equivalent to

$$-v_t-\frac{\sigma^2}{2}(s^2v_{ss}+\gamma_*v)^++\frac{\sigma^2}{2}\gamma_*v=0\ .$$

Since, for any real number ξ,

$$(\xi)^+=\sup_{0\leq a\leq 1}\ \{\ a^2\xi\ \}\ ,$$

$$-v_t+\inf_{0\leq a\leq 1}\{-\frac{\sigma^2a^2}{2}s^2v_{ss}+\frac{\sigma\gamma_*}{2}(1-a^2)v\}=0\ .$$

We recognize the above equation as the dynamic programming equation for the following stochastic optimal control problem:

$$v(t,s)=\sup_{0\leq a(\cdot)\leq 1}E[\exp(-\int_t^T\frac{\sigma^2\gamma_*}{2}(1-a^2(u))du)G(S^a_{t,s}(T))]\ ,$$

where

$$dS^a(u)=\sigma a(u)S^a(u)dW(u)\ .$$

3.4.4 Up and Out European Call Option

This is a path-dependent option

$$\hat{G}(S_{t,s}(\cdot))=\begin{cases}(S_{t,s}(T)-K)^+, & \text{if } \max_{t\leq u\leq T}S_{t,s}(u):=M(T)<B\ ,\\ 0, & M(T)\geq B\ .\end{cases}$$

We can also handle this with PDE techniques. Consider the bound

$$-\gamma_*v\leq s^2v_{ss}\leq\gamma^*v\ .$$

The upper bound describes the modified final data $\hat{G}$:

(i) If

$$s^* := \frac{U}{U-1} K < B \quad (\text{with } U = \frac{1+\sqrt{1+4\gamma^*}}{2}) \ ,$$

then $\hat{G}$ is as in the usual Call Option case.

(ii) If, however, $s^* \geq B$, then

$$\hat{G}(s) = (K-B)(\frac{s}{B})^U \ .$$

We claim that at the lateral boundary $s = B$, the lower gamma bound saturates and

$$s^2 v_{ss}(t,B) = -\gamma_* v(t,B) \ .$$

Using the equation, we formally guess that

$$-v_t(t,B) = \frac{\sigma^2}{2} s^2 v_{ss}(t,B) = -\gamma_* \frac{\sigma^2}{2} v(t,B) \ .$$

We solve this ODE to obtain

$$v(t,B) = (K-B) \ \exp(-\frac{\gamma_* \ \sigma^2}{2} \ (T-t)) \quad \forall t \leq T \ . \tag{3.4.15}$$

Lemma 3.4.1 *The minimal super-replicating cost for the up and out call option is the unique (smooth) solution of*

$$-v_t - \frac{\sigma^2}{2} s^2 v_{ss} = 0, \quad \forall t < T, \ 0 < s < B \ ,$$

$$v(T,s) = \hat{G}(s), \quad \forall 0 \leq s \leq B \ .$$

together with (3.4.15). *In particular,*

$$v(t,s) = E\{G(S_{t,s}(T))\chi_{\{\theta_{t,s} \geq T\}} + (K-B)e^{-\frac{\gamma_* \sigma^2}{2}(t-\theta_{t,s})}\chi_{\{\theta_{t,s} < T\}}\} \ ,$$

where

$$\theta_{t,s} := \inf\{u : \ S_{t,s}(u) = B\} \ .$$

Proof: It is clear that

$$s^2 v_{ss}(t,B) = -\gamma_* v(t,B), \text{ and } s^2 v_{ss}(T,s) \geq 0 \ .$$

Set

$$w(t,s) := s^2 v_{ss}(t,s) + \gamma_* v(t,s) \ .$$

Then,

$$-w_t - \frac{\sigma^2}{2} s^2 w_{ss} = 0, \quad \forall t < T, \ 0 < s < B \ ,$$
$$w(T,s) > 0 \ ,$$
$$w(t,B) = 0 \ .$$

Hence, $w \geq 0$. Similarly set

$$z(t,s) := s^2 v_{ss}(t,s) - \gamma^* v(t,s) \ .$$

Then,

$$z(T,s) \leq, \quad z(t,B) < 0 \ .$$

Also,

$$-z_t - \frac{\sigma^2}{2} s^2 z_{ss} = 0 \ .$$

Hence, $z \leq 0$. Therefore, v solves $\hat{H}(v_t, s^2 v_{ss}, v) = 0$. So by the uniqueness v is the super-replication cost.

□

3.5 Guess for The Dual Formulation

As it was done for the portfolio constraint, using duality is another possible approach to super-replication is also available. We refer to the lecture notes of Rogers [16] for this method and the relevant references. However, the dual approach has not yet been successfully applied to the gamma problem. Here we describe a possible dual problem based on the results obtained through dynamic programming.

Let us first consider the upper bound case

$$s^2 v_{ss} \leq \gamma^* \ .$$

Then the dynamic programming equation is

$$\min\{-v_t - \frac{\sigma^2}{2} s^2 v_{ss};\ -s^2 v_{ss} + \gamma^*\} = 0\ .$$

We rewrite this as

$$-v_t + \inf_{b \geq 1}\{-\frac{\sigma^2}{2} b^2 s^2 v_{ss} + \frac{\sigma^2}{2}\gamma^*(b^2 - 1)\} = 0\ .$$

The above equation is the dynamic programming equation of the following optimal control problem,

$$v(t,s) := \sup_{b(\cdot)\geq 1} E\left[\ -\frac{\sigma^2}{2}\gamma^*\ \int_t^T (b^2(u) - 1)du + G(S^b_{t,s}(T))\ \right]\ ,$$

$$dS^b_{t,s}(u) = \sigma b(u) S^b_{t,s}(u) dW(u)\ .$$

Note the change in the diffusion coefficient of the stock price process.

If we consider,

$$-\ \gamma_* \leq s^2 v_{ss}\ ,$$

same argument yields

$$v(t,s) = \sup_{0\leq a(\cdot)\leq 1} E\left[\ -\frac{\sigma^2}{2}\gamma_*\ \int_t^T (1 - a^2(u))du + G(S^a_{t,s}(T))\ \right]\ .$$

Now consider the full constraint,

$$-\gamma_* \leq s^2 v_{ss} \leq \gamma^*\ .$$

The equation is

$$\sup_{Q\geq 0}\ \min\{-v_t - \frac{\sigma^2}{2}(s^2 v_{ss} + Q);\ -(s^2 v_{ss} + Q) + \gamma^*;\ (s^2 v_{ss} + Q) + \gamma_*\} = 0\ .$$

We rewrite it as

$$-v_t + \inf_{b\geq 1\geq a\geq 0}\{-\frac{\sigma^2}{2} s^2 a^2 b^2 v_{ss} + \frac{\sigma^2}{2}\gamma^*(b^2 - 1) + \frac{\sigma^2}{2}\gamma_*(1 - a^2)\} = 0\ .$$

Hence,

$$v(t,s) = \sup_{b(\cdot)\geq 1, 0\leq a(\cdot)\leq 1} E\left[\ -\frac{\sigma^2}{2}\ \int_t^T [\gamma^*(b(u)^2 - 1) + \gamma_*(1 - a^2(u))]du + G(S^{a,b}_{t,s}(T))\ \right]\ ,$$

$$dS^{a,b}(u) = \sigma a(u) b(u) S^{a,b}(u) dW(u) \ .$$

It is *open* to prove this by direct convex analysis methods. We finish by observing that if

$$-\gamma_* v \leq s^2 v_{ss} \leq \gamma^* v \ ,$$

then

$$v(t,s) = \sup_{b(\cdot)\geq 1, 0\leq a(\cdot)\leq 1} E\left[\ e^{\int_t^T -\frac{\sigma^2}{2}[\gamma^*(b(u)^2-1)+\gamma_*(1-a^2(u))] \ du} \ G(S_{t,s}^{a,b}(T)) \ \right] \ .$$

All approaches to duality (see Rogers' lecture notes [16]) yield expressions of the form

$$v(t,s) = \sup E[\ B \ Y \ ] \ ,$$

where $B = G(S(T))$ in our examples. However, in above examples $S(\cdot)$ needs to be modified in a way that is not absolutely continuous with respect to P.

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