Subsequential scaling limits of simple random walk on the two-dimensional uniform spanning tree

ASPECTS OF RANDOM WALKS
DURHAM UNIVERSITY, 31 MARCH – 3 APRIL 2014

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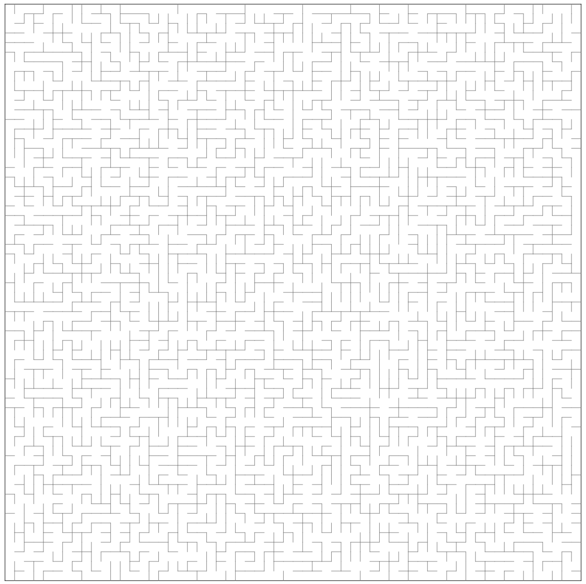
UNIFORM SPANNING TREE IN TWO DIMENSIONS



Let $\Lambda_n := [-n,n]^2 \cap \mathbb{Z}^2$.

A subgraph of the lattice is a **spanning tree** of Λ_n if it connects all vertices and has no cycles.

Let $\mathcal{U}^{(n)}$ be a spanning tree of Λ_n selected uniformly at random from all possibilities.

The UST on $\mathbb{Z}^2$, $\mathcal{U}$, is then the local limit of $\mathcal{U}^{(n)}$.
NB. Wired/free boundary conditions unimportant.

Almost-surely, $\mathcal{U}$ is a spanning tree of $\mathbb{Z}^2$. (Forest for $d > 4$.)

[Aldous, Benjamini, Broder, Häggström, Kirchoff, Lyons, Pemantle, Peres, Schramm, Wilson,. . .]

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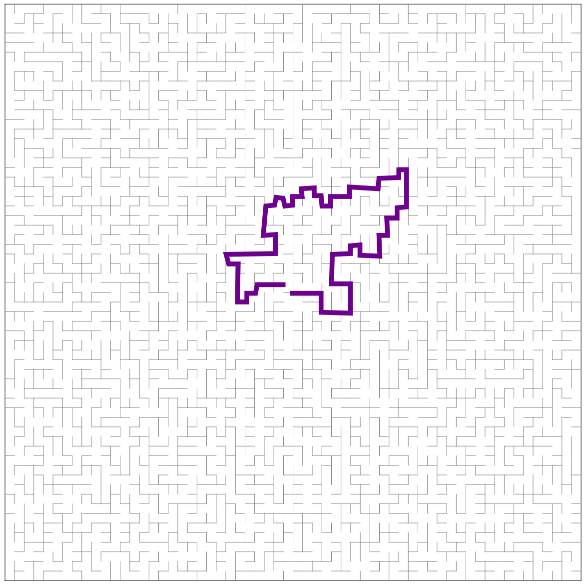

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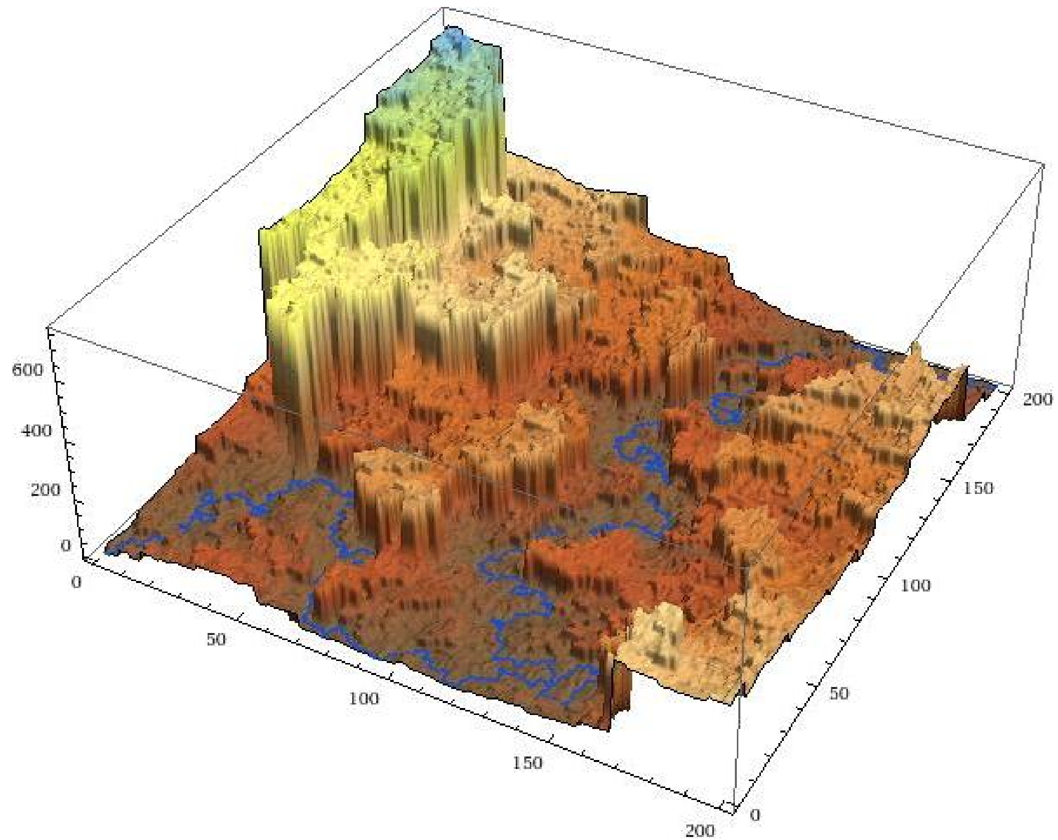


The distances in the tree to the path between opposite corners in a uniform spanning tree in a 200×200 grid.

Picture: Lyons/Peres: Probability on trees and networks

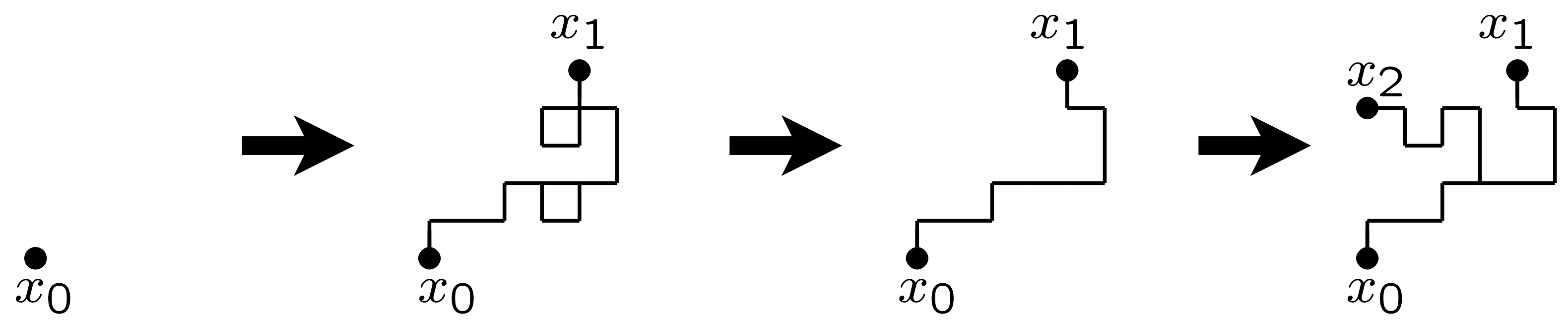
WILSON'S ALGORITHM ON $\mathbb{Z}^2$

Let $x_0 = 0, x_1, x_2, \ldots$ be an enumeration of $\mathbb{Z}^2$.

Let $\mathcal{U}(0)$ be the graph tree consisting of the single vertex x_0.

Given $\mathcal{U}(k-1)$ for some $k \geq 1$, define $\mathcal{U}(k)$ to be the union of $\mathcal{U}(k-1)$ and the loop-erased random walk (LERW) path run from x_k to $\mathcal{U}(k-1)$.

The UST $\mathcal{U}$ is then the local limit of $\mathcal{U}(k)$.

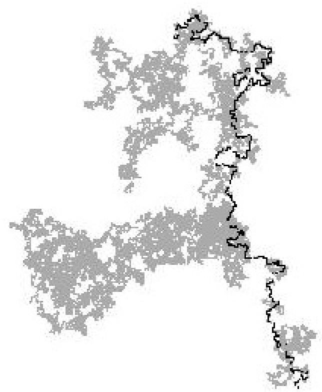


LERW SCALING IN $\mathbb{Z}^d$

Consider LERW as a process $(L_n)_{n\geq 0}$ (assume original random walk is transient).

In $\mathbb{Z}^d$, $d \geq 5$, L rescales diffusively to Brownian motion [Lawler].

In $\mathbb{Z}^4$, with logarithmic corrections rescales to Brownian motion [Lawler].

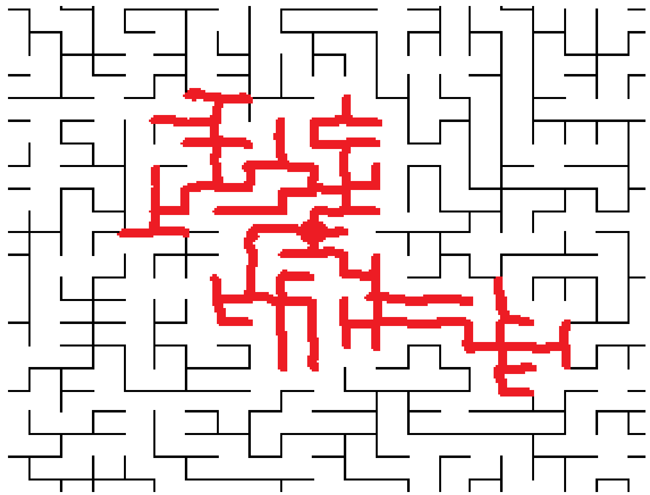


Picture: Ariel Yadin

In $\mathbb{Z}^3$, $\{L_n : n \in [0,\tau]\}$ has a scaling limit [Kozma, Shiraishi].

In $\mathbb{Z}^2$, $\{L_n : n \in [0,\tau]\}$ has SLE(2) scaling limit [Lawler/Schramm/Werner]. Growth exponent is 5/4 [Kenyon, Masson, Lawler].

VOLUME AND RESISTANCE ESTIMATES [BARLOW/MASSON]



With high probability,

$$B_E(x, \lambda^{-1} R) \subseteq B_{\mathcal{U}}(x, R^{5/4}) \subseteq B_E(x, \lambda R),$$

as $R \to \infty$ then $\lambda \to \infty$.

It follows that with high probability,

$$\mu_{\mathcal{U}}\,(B_{\mathcal{U}}(x, R)) \asymp R^{8/5}.$$

Also with high probability,

$$\text{Resistance}(x, B_{\mathcal{U}}(x, R)^c) \asymp R.$$

Implies exit time for intrinsic ball radius R is $R^{13/5}$, also heat kernel bounds $p^{\mathcal{U}}_{2n}(0,0) \asymp n^{-8/13}$. cf. [Barlow/Jarai/Kumagai/Slade]

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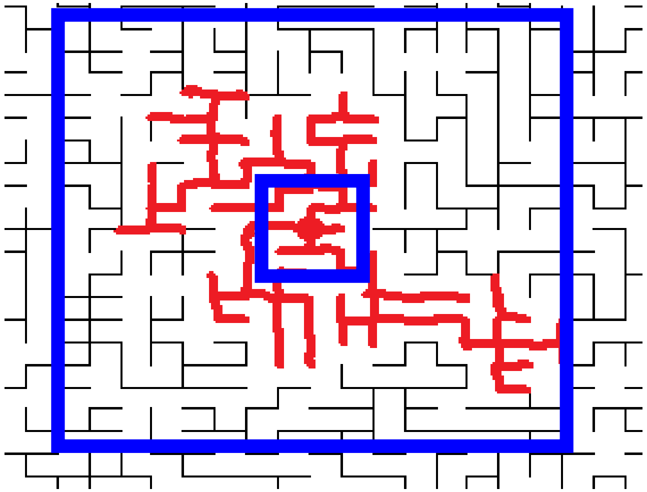

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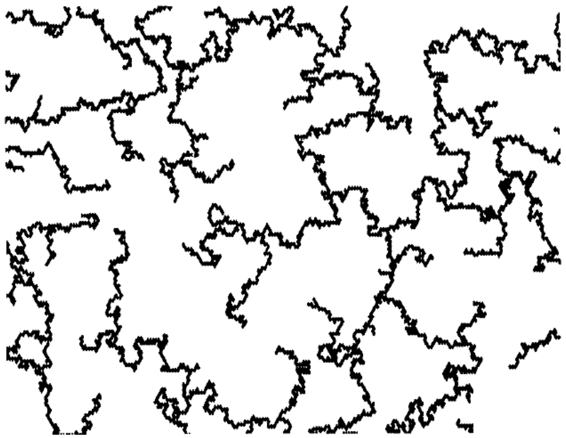
How about scaling limit for UST? And for SRW on the UST?

UST SCALING [SCHRAMM]

Consider $\mathcal{U}$ as an ensemble of paths:

$$\mathfrak{U} = \left\{(a, b, \pi_{ab}) :\ a, b \in \mathbb{Z}^2\right\},$$

where π_{ab} is the unique arc connecting a and b in $\mathcal{U}$, as an element of the compact space $\mathcal{H}(\dot{\mathbb{R}}^2 \times \dot{\mathbb{R}}^2 \times \mathcal{H}(\dot{\mathbb{R}}^2))$,
cf. [Aizenman/Burchard/Newman/Wilson].



Picture: Oded Schramm

Scaling limit $\mathfrak{T}$ almost-surely satisfies:

- each pair $a, b \in \dot{\mathbb{R}}^2$ connected by a path;
- if $a \neq b$, then this path is simple;
- if $a = b$, then this path is a point or a simple loop;
- the trunk, $\cup_{\mathfrak{T}} \pi_{ab} \backslash \{a, b\}$, is a dense topological tree with degree at most 3.

ISSUE: This topology does not carry information about intrinsic distance, volume, or resistance.

GENERALISED GROMOV-HAUSDORFF TOPOLOGY (cf. [GROMOV, LE GALL/DUQUESNE])

Define $\mathbb{T}$ to be the collection of measured, rooted, spatial trees, i.e.

$$(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}),$$

where:

- $(\mathcal{T}, d_{\mathcal{T}})$ is a complete and locally compact real tree;
- $\mu_{\mathcal{T}}$ is a locally finite Borel measure on $(\mathcal{T}, d_{\mathcal{T}})$;
- $\phi_{\mathcal{T}}$ is a continuous map from $(\mathcal{T}, d_{\mathcal{T}})$ into $\mathbb{R}^2$;
- $\rho_{\mathcal{T}}$ is a distinguished vertex in $\mathcal{T}$.

On $\mathbb{T}_c$ (compact trees only), define a distance Δ_c by

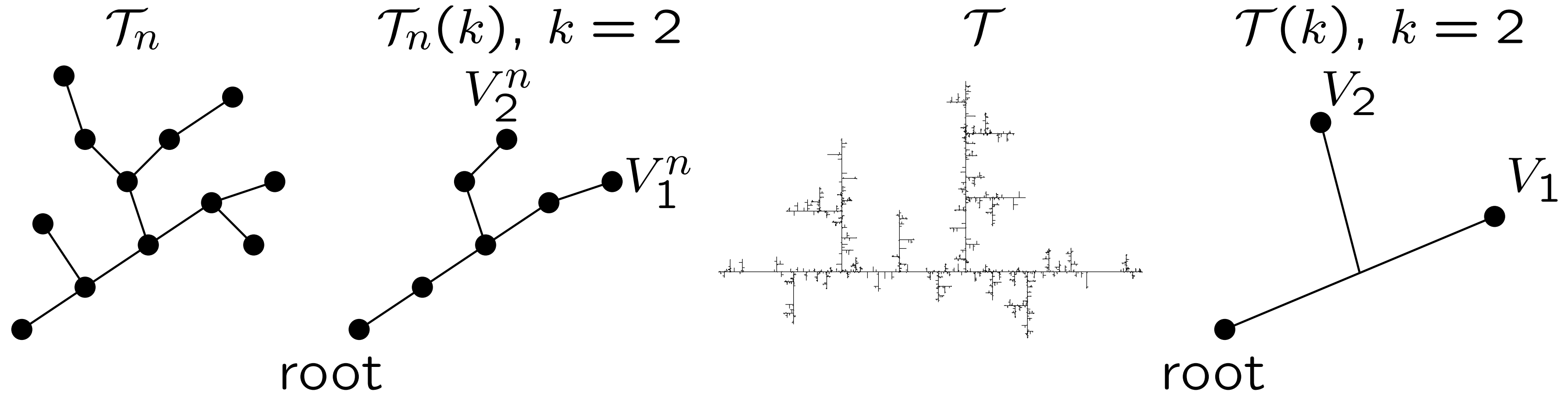
$$\inf_{\substack{Z,\psi,\psi',\mathcal{C}: \\ (\rho_{\mathcal{T}},\rho'_{\mathcal{T}})\in\mathcal{C}}} \left\{ d_P^Z(\mu_{\mathcal{T}} \circ \psi^{-1}, \mu'_{\mathcal{T}} \circ \psi'^{-1}) + \sup_{(x,x')\in\mathcal{C}} \Big(d_Z(\psi(x), \psi'(x')) + \big| \phi_{\mathcal{T}}(x) - \phi'_{\mathcal{T}}(x') \big| \Big) \right\}.$$

Can be extended to locally compact case.

F.D.D. AND TIGHTNESS

The convergence $\Delta_c(\mathcal{T}_n, \mathcal{T}) \to 0$ is equivalent to:

1. Convergence of finite dimensional distributions:



For suitably chosen minimal subtrees spanning root and k vertices, establish $\mathcal{T}_n(k) \to \mathcal{T}(k)$, where $\mathcal{T}(k) \to \mathcal{T}$.

2. Tightness: Show that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \Delta_c(\mathcal{T}_n, \mathcal{T}_n(k)) = 0.$$

TIGHTNESS OF UST

Theorem. If $\mathbf{P}_\delta$ is the law of the measured, rooted spatial tree

$$\left(\mathcal{U}, \delta^{5/4} d_{\mathcal{U}}, \delta^2 \mu_{\mathcal{U}}(\cdot), \delta \phi_{\mathcal{U}}, 0\right)$$

under $\mathbf{P}$, then the collection $(\mathbf{P}_\delta)_{\delta \in (0,1)}$ is tight in $\mathcal{M}_1(\mathbb{T})$.

Proof involves:

- strengthening estimates of [Barlow/Masson],
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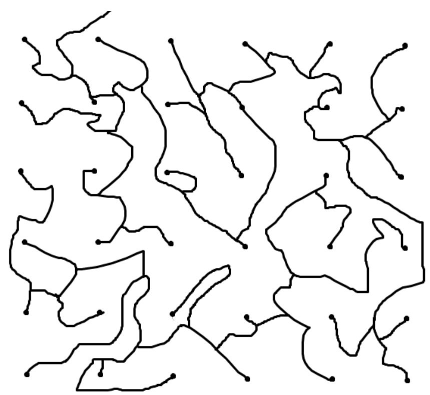

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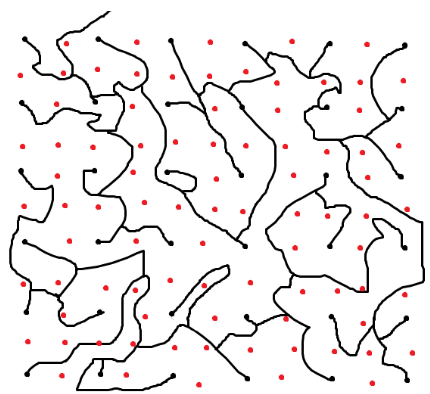

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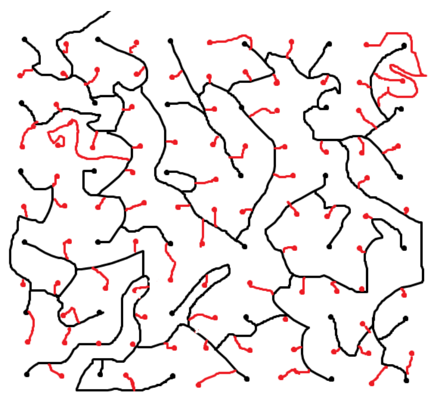


UST LIMIT PROPERTIES

If $\tilde{\mathbf{P}}$ is a subsequential limit of $(\mathbf{P}_\delta)_{\delta\in(0,1)}$, then for $\tilde{\mathbf{P}}$-a.e. $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$ it holds that:

(a) (i) the Hausdorff dimension of $(\mathcal{T}, d_{\mathcal{T}})$ is given by

$$d_f := \frac{8}{5};$$

(ii) $(\mathcal{T}, d_{\mathcal{T}})$ has precisely one end at infinity;

(b) (i) $\mu_{\mathcal{T}}$ is non-atomic and supported on the leaves of $\mathcal{T}$, i.e. $\mu_{\mathcal{T}}(\mathcal{T}^o) = 0$, where $\mathcal{T}^o := \mathcal{T} \backslash \{x \in \mathcal{T} : \deg_{\mathcal{T}}(x) = 1\}$;

(ii) given $R > 0$, uniformly for $x \in B_{\mathcal{T}}(\rho_{\mathcal{T}}, R)$ and $r \in (0, r_0)$,

$$c_1 r^{d_f} (\log r^{-1})^{-80} \leq \mu_{\mathcal{T}}(B_{\mathcal{T}}(x, r)) \leq c_2 r^{d_f} (\log r^{-1})^{80};$$

(iii) uniformly for $r \in (0, r_0)$,

$$c_1 r^{d_f} (\log\log r^{-1})^{-9} \leq \mu_{\mathcal{T}}(B_{\mathcal{T}}(\rho, r)) \leq c_2 r^{d_f} (\log\log r^{-1})^{3};$$

(c) (i) the restriction of the continuous map $\phi_{\mathcal{T}} : \mathcal{T} \to \mathbb{R}^2$ to $\mathcal{T}^o$ is a homeomorphism between this set and its image $\phi_{\mathcal{T}}(\mathcal{T}^o)$, which is dense in $\mathbb{R}^2$;

(ii) $\max_{x \in \mathcal{T}} \deg_{\mathcal{T}}(x) = 3 = \max_{x \in \mathbb{R}^2} |\phi_{\mathcal{T}}^{-1}(x)|$;

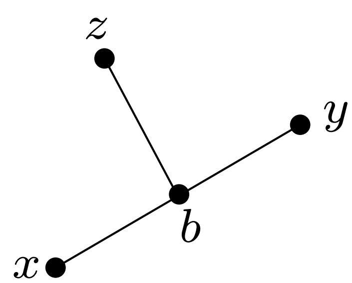
(iii) $\mu_{\mathcal{T}} = \mathcal{L} \circ \phi_{\mathcal{T}}$.

BROWNIAN MOTION ON REAL TREES

Given an element of $\mathbb{T}$ such that $\mu^{\mathcal{T}}$ has full support, then it is possible to define a 'Brownian motion' $X^{\mathcal{T}} = (X_t^{\mathcal{T}})_{t\geq 0}$ on $(\mathcal{T}, d_{\mathcal{T}}, \mu^{\mathcal{T}})$ [Krebs, Kigami, Athreya/Eckhoff/Winter].

- Strong Markov diffusion.
- Reversible, invariant measure $\mu^{\mathcal{T}}$.
- For $x, y, z \in \mathcal{T}$,

$$P_z^{\mathcal{T}}(\tau_x < \tau_y) = \frac{d_{\mathcal{T}}(b(x,y,z),y)}{d_{\mathcal{T}}(x,y)}.$$



- Mean occupation density when started at x and killed at y,

$$2d_{\mathcal{T}}(b(x,y,z),y)\mu^{\mathcal{T}}(dz).$$

$\mathbb{T}^*$: $\mu^{\mathcal{T}}$ is non-atomic, supported on leaves, and, for some $\kappa > 0$,

$$\liminf_{r\to 0} \frac{\inf_{x\in\mathcal{T}} \mu_{\mathcal{T}}(B_{\mathcal{T}}(x,r))}{r^\kappa} > 0.$$

On $\mathbb{T}^*$, can further define jointly continuous local times.

CONVERGENCE OF SRW (cf. [C.])

Let $(T_n)_{n\geq 1}$ be a sequence of finite graph trees, and X^{T_n} the SRW on T_n.

Suppose that there exist null sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, $(c_n)_{n\geq 1}$ with $b_n = o(a_n)$ such that

$$\left(T_n, a_n d_{T_n}, b_n \mu_{T_n}, c_n \phi_{T_n}, \rho_{T_n}\right) \rightarrow (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$$

in $(\mathbb{T}_c, \Delta_c)$, where $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$ is an element of $\mathbb{T}_c^*$. Let $X^{\mathcal{T}}$ be Brownian motion on $\mathcal{T}$, then

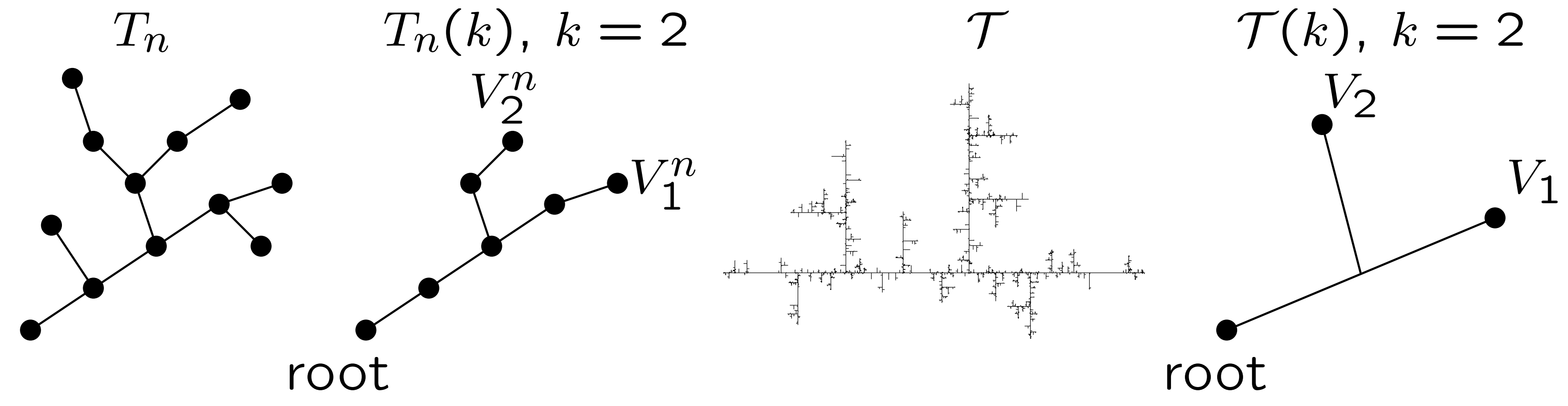
$$\left(c_n \phi_{T_n}\left(X^{T_n}_{t/a_n b_n}\right)\right)_{t\geq 0} \rightarrow \left(\phi_{\mathcal{T}}\left(X^{\mathcal{T}}_t\right)\right)_{t\geq 0}$$

in distribution in $C(\mathbb{R}_+, \mathbb{R}^2)$, where we assume $X_0^{T_n} = \rho_{T_n}$ for each n, and also $X_0^{\mathcal{T}} = \rho_{\mathcal{T}}$.

Can also extend to locally compact case.

PROOF IDEA

We can assume that $a_n T_n(k) \to \mathcal{T}(k)$ for suitable subtrees.



Step 1: Show processes on graph subtrees converge for each k.

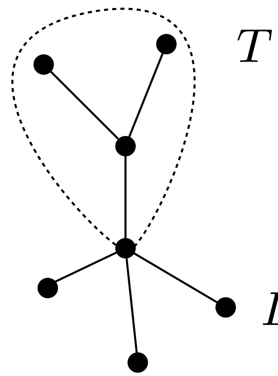
Step 2: Time-change using projected measures $\mu_{n,k} = \mu_{T_n} \circ \pi_{n,k}^{-1}$.

Step 3: Show these are close to processes of interest as $k \to \infty$.

ELEMENTARY SRW IDENTITY

Let T be a rooted graph tree, and attach D extra vertices at its root, each by a single edge.

e.g.



If $\alpha(T, D)$ is the expected time for a simple random walk started from the root to hit one of the extra vertices, then

$$\alpha(T, D) = \frac{2\#V(T) - 2 + D}{D}.$$

In particular, if $D = 2$, then

$$\alpha(T, D) = \#V(T).$$

LIMITING PROCESS FOR SRW ON UST

Suppose $(\mathbf{P}_{\delta_i})_{i\geq 1}$, the laws of

$$\left(\mathcal{U}, \gamma(\delta_{n_i}) d_{\mathcal{U}}, \delta_{n_i}^2 \mu_{\mathcal{U}}, \delta_{n_i} \phi_{\mathcal{U}}, 0\right),$$

form a convergent sequence with limit $\tilde{\mathbf{P}}$.

If $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$ has law $\tilde{\mathbf{P}}$, then the annealed law of

$$\left(\phi_{\mathcal{T}}(X_t^{\mathcal{T}})\right)_{t\geq 0},$$

where $X^{\mathcal{T}}$ is Brownian motion on $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}})$ started from $\rho_{\mathcal{T}}$, i.e.

$$\tilde{\mathbb{P}}(\cdot) := \int P_{\rho_{\mathcal{T}}}^{\mathcal{T}} \circ \phi_{\mathcal{T}}^{-1}(\cdot) \tilde{\mathbf{P}}(d\mathcal{T}),$$

is a well-defined probability measure on $C(\mathbb{R}_+, \mathbb{R}^2)$.

CONVERGENCE OF SRW ON UST

Suppose $(\mathbf{P}_{\delta_i})_{i\geq 1}$, the laws of

$$\left(\mathcal{U}, \delta_i^{5/4} d_{\mathcal{U}}, \delta_i^2 \mu_{\mathcal{U}}, \delta_i \phi_{\mathcal{U}}, 0\right),$$

form a convergent sequence with limit $\tilde{\mathbf{P}}$.

Let $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}) \sim \tilde{\mathbf{P}}$.

It is then the case that $\mathbb{P}_{\delta_i}$, the annealed laws of

$$\left(\delta_i X^{\mathcal{U}}_{\delta_i^{-13/4} t}\right)_{t\geq 0},$$

converge to $\tilde{\mathbb{P}}$, the annealed law of

$$\left(\phi_{\mathcal{T}}(X_t^{\mathcal{T}})\right)_{t\geq 0},$$

as probability measures on $C(\mathbb{R}_+, \mathbb{R}^2)$.

HEAT KERNEL ESTIMATES FOR SRW LIMIT

Let $R > 0$. For $\tilde{\mathbf{P}}$-a.e. realisation of $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$, there exist random constants $c_1, c_2, c_3, c_4, t_0 \in (0,\infty)$ and deterministic constants $\theta_1, \theta_2, \theta_3, \theta_4 \in (0,\infty)$ such that the heat kernel associated with the process $X^{\mathcal{T}}$ satisfies:

$$p_t^{\mathcal{T}}(x,y) \leq c_1 t^{-8/13} \ell(t^{-1})^{\theta_1} \exp\left\{ -c_2 \left(\frac{d_{\mathcal{T}}(x,y)^{13/5}}{t} \right)^{5/8} \ell(d_{\mathcal{T}}(x,y)/t)^{-\theta_2} \right\},$$

$$p_t^{\mathcal{T}}(x,y) \geq c_3 t^{-8/13} \ell(t^{-1})^{-\theta_3} \exp\left\{ -c_4 \left(\frac{d_{\mathcal{T}}(x,y)^{13/5}}{t} \right)^{5/8} \ell(d_{\mathcal{T}}(x,y)/t)^{\theta_4} \right\},$$

for all $x, y \in B_{\mathcal{T}}(\rho_{\mathcal{T}}, R)$, $t \in (0, t_0)$, where $\ell(x) := 1 \vee \log x$.

RANGE OF SRW ON UST

After 5,000 and 50,000 steps: