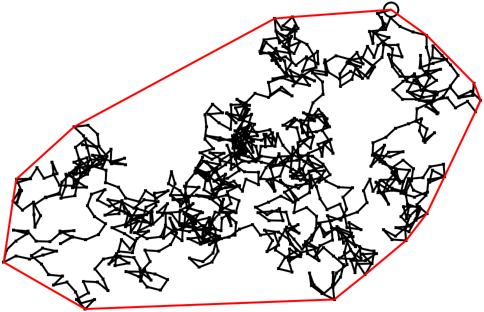
Convex hulls of random walks



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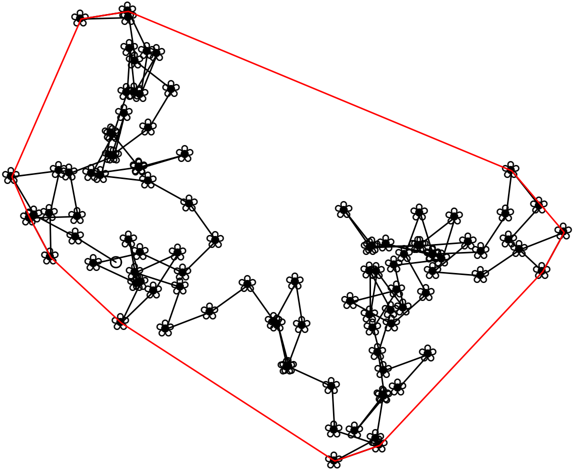
Introduction

On each of n unsteady steps, a drunken gardener drops a seed. Once the flowers have bloomed, what is the minimum length of fencing needed to enclose the garden?



Introduction

Let $Z_1, Z_2, \ldots$ be independent, identically distributed random vectors in $\mathbb{R}^2$.

The Z_i will be the increments of the planar random walk S_n, $n \geq 0$, defined by $S_0 = 0$ (the origin in $\mathbb{R}^2$) and

$$S_n = \sum_{i=1}^{n} Z_i.$$

We are interested in asymptotic properties of the convex hull $\text{hull}(S_0, \ldots, S_n)$.

In particular, the $n \to \infty$ limit behaviour of the random variables

$$L_n = \text{the perimeter length of hull}(S_0, \ldots, S_n).$$

Introduction

Standing assumption: $\mathbb{E}(\|Z_1\|^2) < \infty$ (two moments).

There is going to be a clear distinction between the zero drift case ($\mathbb{E}Z_1 = 0$) and the non-zero drift case ($\|\mathbb{E}Z_1\| > 0$).

For example, under mild conditions:

- the zero-drift walk is recurrent and the convex hull tends to the whole of $\mathbb{R}^2$;
- the walk with drift is transient with a limiting direction and the convex hull sits inside some arbitrarily narrow wedge.

We will look at both cases in this talk. First, we give a brief summary of some history.

1. Introduction
2. Background
3. Zero drift and Brownian scaling limit
4. Non-zero drift and central limit theorem
5. Concluding remarks

Some history

Spitzer & Widom (1961) and Baxter (1961) showed that

$$\mathbb{E} L_n = 2 \sum_{k=1}^{n} \frac{1}{k} \mathbb{E} \| S_k \| .$$

So, under mild conditions:

- the zero-drift case has $\mathbb{E} L_n \asymp \sqrt{n}$;
- the case with drift has $\mathbb{E} L_n \asymp n$.

Snyder & Steele (1993) showed that

$$\frac{1}{n} \mathbb{V}\mathrm{ar}(L_n) \leq \frac{\pi^2}{2} \left(\mathbb{E} \| Z_1 \|^2 - \| \mathbb{E} Z_1 \|^2 \right) . \tag{1}$$

Snyder & Steele deduced from (1) the strong law

$$\lim_{n \to \infty} n^{-1} L_n = 2 \| \mathbb{E} Z_1 \| , \text{ a.s.}$$

Some questions

The work of Snyder & Steele raised some natural questions.

- Is n the correct order for $\mathbb{V}\text{ar}(L_n)$?
- Is there a distributional limit theorem for L_n?
- If so, is the limit distribution normal?

The answers to these questions turn out be be essentially

- yes, yes, no in the zero drift case, and
- yes, yes, yes in the non-zero drift case,

excluding some degenerate cases.

First, we have a quick look at some simulation evidence.

Some simulations

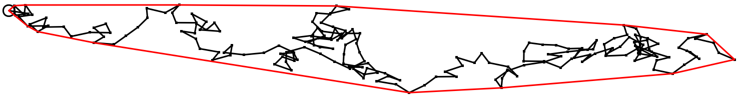
As a concrete example, let Z_1 be distributed as $(\mu, 0) + \mathbf{e}_\Theta$, where

- $\mathbf{e}_\theta = (\cos\theta, \sin\theta)$ is the unit vector in direction θ;
- Θ is a uniform random variable on $[0, 2\pi)$.

So $\mathbb{E} Z_1 = (\mu, 0)$, $\|\mathbb{E} Z_1\| = \mu$.

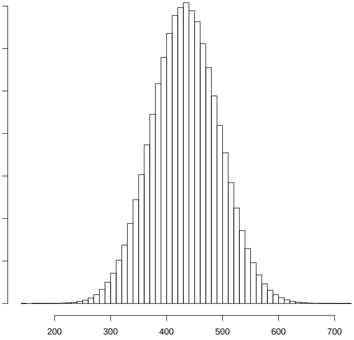
The case $\mu = 0$ is the Pearson–Rayleigh random walk.

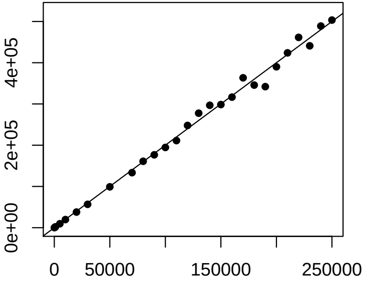
Picture for $\mu = 0.2$:



Some simulations: non-zero drift

- Left: Histogram of L_n samples for $n = 10^4$; 10^6 simulations.
- Right: Plot of point estimates for $y = \mathbb{V}\mathrm{ar}(L_n)$ against $x = n$; also shown is the line $y = 2x$.



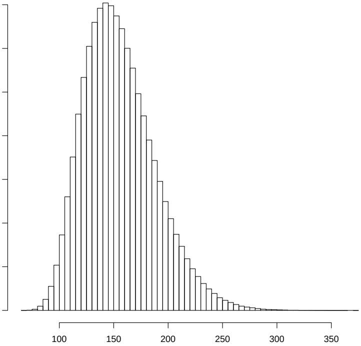


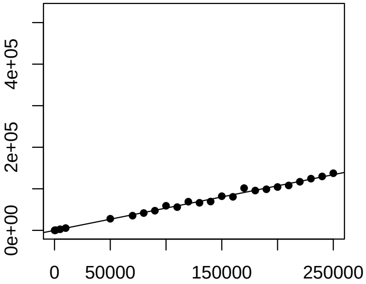
Simulations suggest:

$$n^{-1}\mathbb{V}\mathrm{ar}(L_n) \to \text{const. (2 in this case!)}; \quad L_n \text{ satisfies a CLT.}$$

Some simulations: zero drift

- Left: Histogram of L_n samples for $n = 10^4$; 10^6 simulations.
- Right: Plot of point estimates for $y = \mathbb{V}\mathrm{ar}(L_n)$ against $x = n$; also shown is the line $y = 0.536x$.





Simulations suggest:

$$n^{-1}\mathbb{V}\mathrm{ar}(L_n) \to \text{const.} \approx 0.536; \quad L_n \text{ non-Gaussian limit.}$$

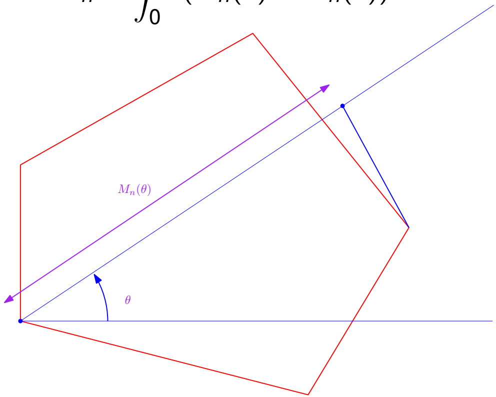
First tool: Cauchy formula

Let $\mathbf{e}_\theta = (\cos\theta, \sin\theta)$, unit vector in direction θ. Set

$$M_n(\theta) = \max_{0 \le k \le n} (S_k \cdot \mathbf{e}_\theta), \quad m_n(\theta) = \min_{0 \le k \le n} (S_k \cdot \mathbf{e}_\theta).$$

Cauchy's perimeter formula from convex geometry:

$$L_n = \int_0^\pi (M_n(\theta) - m_n(\theta))\, \mathrm{d}\theta.$$



First tool: Cauchy formula

$$L_n = \int_0^{\pi} (M_n(\theta) - m_n(\theta))\,\mathrm{d}\theta.$$

A first consequence: classical fluctuation theory for random walk on $\mathbb{R}$ gives

$$\mathbb{E}M_n(\theta) = \sum_{k=1}^{n} k^{-1}\mathbb{E}[(S_k \cdot \mathbf{e}_\theta)^+],$$

a formula attributed variously to Kac, Hunt, Dyson, and Chung, and which can be proved combinatorially, or analytically as a consequence of the Spitzer–Baxter fluctuation theory identities. Then

$$\mathbb{E}L_n = \sum_{k=1}^{n} k^{-1}\mathbb{E}\int_0^{\pi} |S_k \cdot \mathbf{e}_\theta|\mathrm{d}\theta = 2\sum_{k=1}^{n} k^{-1}\mathbb{E}\|S_k\|,$$

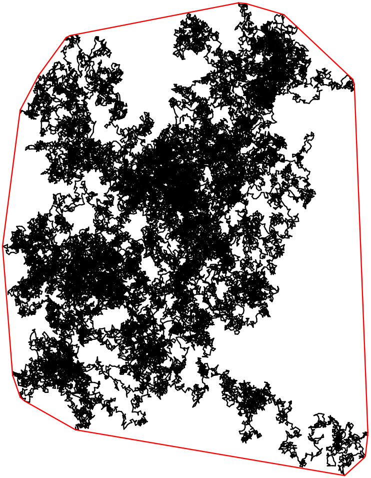
which is the Spitzer–Widom formula.

Zero drift case

Suppose $\mathbb{E}Z_1 = 0$. The random walk has Brownian motion as its scaling limit.

So one would expect that the convex hull of the random walk is described in the limit by the convex hull of Brownian motion. The latter was studied by Lévy; more recently by El Bachir (1983) and others.

We need to know a little about convex hulls of continuous paths, and need to set things up on the right space(s).



Paths and hulls

Consider continuous $f : [0, T] \to \mathbb{R}^d$ with $f(0) = 0$; say $f \in \mathcal{C}_d^0$. (T is not very important—enough to take $T \equiv 1$.)

With the supremum norm $\rho_\infty(f, g) = \sup_x \|f(x) - g(x)\|$ we get a metric space $(\mathcal{C}_d^0, \rho_\infty)$.

The path segment ($\equiv$ interval image) $f[0, t] = \{f(s) : s \in [0, t]\}$ is compact. $\Longrightarrow$ hull($f[0, t]$) is compact (by a theorem of Carathéodory).

That is, hull($f[0, t]$) is an element of the metric space $(\mathcal{K}_d^0, \rho_H)$ of compact convex subsets of $\mathbb{R}^d$ containing 0, with the Hausdorff metric.

Paths and hulls

Metric space $(\mathcal{K}_d^0, \rho_H)$ of compact convex subsets of $\mathbb{R}^d$ containing 0, with the Hausdorff metric.

Given $A \in \mathcal{K}_d^0$ and $r > 0$, let $A^r := \{x \in \mathbb{R}^d : \rho(x, A) \leq r\}$.

For $A, B \in \mathcal{K}_d^0$,

$$\rho_H(A, B) \leq r \quad \Leftrightarrow \quad A \subseteq B^r \text{ and } B \subseteq A^r.$$

Lemma 1

For each t, the map $f \mapsto hull(f[0, t])$ is a continuous function from $(\mathcal{C}_d^0, \rho_\infty)$ to $(\mathcal{K}_d^0, \rho_H)$.

Scaling limit

Given random walk $S_n = \sum_{i=1}^{n} Z_i$, define

$$X_n(t) := n^{-1/2} \left(S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) \left(S_{\lfloor nt \rfloor + 1} - S_{\lfloor nt \rfloor} \right) \right).$$

So for each n, $X_n \in \mathcal{C}_d^0$; $X_n(0) = 0$ and $X_n(1) = n^{-1/2} S_n$.
Let b_t, $t \geq 0$ denote standard Brownian motion on $\mathbb{R}^d$.

Donsker's Theorem

Suppose $\mathbb{E}(\|Z_1\|^2) < \infty$, $\mathbb{E}Z_1 = 0$, *and* $\mathbb{E}(Z_1 Z_1^\top) = \sigma^2 I$, $\sigma^2 > 0$.
Then $X_n/\sigma \Rightarrow b$ *in the sense of weak convergence on* $(\mathcal{C}_d^0, \rho_\infty)$.

Note $\text{hull}(X_n[0,1]) = n^{-1/2} \, \text{hull}(S_0, \ldots, S_n)$. Then with Lemma 1 and the continuous mapping theorem, we get:

Theorem 2

Under the same conditions,
$n^{-1/2}$ *hull*$(S_0, \ldots, S_n) \Rightarrow$ *hull*$(b[0,1])$ *in the sense of weak convergence on* $(\mathcal{K}_d^0, \rho_H)$.

Functionals

Now take $d = 2$. One neat way to define perimeter length of a set $A \in \mathcal{K}_2^0$ is via intrinsic volumes:

$$\mathcal{L}(A) := \lim_{r \downarrow 0} \left(\frac{|A^r| - |A|}{r} \right),$$

where $|\cdot|$ is Lebesgue measure on $\mathbb{R}^2$; the limit exists by the Steiner formula of integral geometry. In particular,

$$\mathcal{L}(A) = \begin{cases} \mathcal{H}_1(\partial A) & \text{if } \operatorname{int}(A) \neq \emptyset \\ 2\mathcal{H}_1(\partial A) & \text{if } \operatorname{int}(A) = \emptyset \end{cases}$$

where $\mathcal{H}_1$ is one-dimensional Hausdorff measure.

Functionals

Lemma 3

The map $A \mapsto \mathcal{L}(A)$ is a continuous function from $(\mathcal{K}_2^0, \rho_H)$ to $(\mathbb{R}_+, \rho)$.

Note $\mathcal{L}(X_n[0,1]) = \mathcal{L}(n^{-1/2}\,\text{hull}(S_0, \ldots, S_n)) = n^{-1/2} L_n$.

Corollary 4

Suppose $\mathbb{E}(\|Z_1\|^2) < \infty$, $\mathbb{E}Z_1 = 0$, and $\mathbb{E}(Z_1 Z_1^\top) = \sigma^2 I$, $\sigma^2 > 0$. Then $n^{-1/2} L_n \xrightarrow{d} \ell_1$, where $\ell_1 = \mathcal{L}(\text{hull}(b[0,1]))$ is the perimeter length of the convex hull of planar Brownian motion run for unit time.

Assuming $\mathbb{E}(\|Z_1\|^{2+\varepsilon}) < \infty$, a uniform integrability argument gives

$$\lim_{n \to \infty} n^{-1} \mathbb{V}\text{ar}(L_n) = \mathbb{V}\text{ar}(\ell_1).$$

Work in progress. We can show $\mathbb{V}\text{ar}(\ell_1) > 0$. We'd like an exact formula. Goldman (1996) manages to do a similar calculation for the planar Brownian bridge, but it is tricky.

Non-zero drift case

Now suppose $\|\mathbb{E}Z_1\| > 0$. Our results are:

Theorem 5

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{V}ar(L_n) = \frac{4\mathbb{E}[((Z_1 - \mathbb{E}Z_1)\cdot\mathbb{E}Z_1)^2]}{\|\mathbb{E}Z_1\|^2} =: s^2 \in [0,\infty).$$

Theorem 6

Suppose that $s^2 > 0$. Then for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{L_n - \mathbb{E}L_n}{\sqrt{\mathbb{V}arL_n}} \le x\right) = \lim_{n\to\infty} \mathbb{P}\left(\frac{L_n - \mathbb{E}L_n}{\sqrt{ns^2}} \le x\right) = \Phi(x),$$

the standard normal distribution function.

Remarks

(i) A little algebra shows $s^2 \leq 4(\mathbb{E}\|Z_1\|^2 - \|\mathbb{E}Z_1\|^2)$.

Compare to the Snyder–Steele upper bound
$n^{-1}\mathbb{V}\text{ar}(L_n) \leq \frac{\pi^2}{2}(\mathbb{E}\|Z_1\|^2 - \|\mathbb{E}Z_1\|^2)$.

I.e., the constant in the Snyder–Steele upper bound is not sharp ($4 < \pi^2/2$).

(ii) $s^2 = 0$ if and only if $Z_1 - \mathbb{E}Z_1$ is a.s. orthogonal to $\mathbb{E}Z_1$.

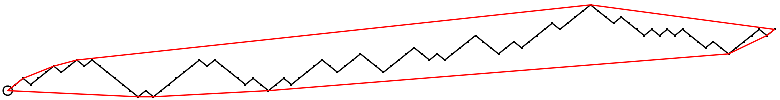
This is the case, for instance, if Z_1 takes values $(1, 1)$ or $(1, -1)$, each with probability $1/2$.

In this case Theorem 5 says that $\mathbb{V}\text{ar}(L_n) = o(n)$.
The Snyder–Steele bound says only that $\mathbb{V}\text{ar}(L_n) \leq \pi^2 n/2$.
Simulations suggest that actually $\mathbb{V}\text{ar}(L_n) = O(\log n)$.

Degenerate example

Z_1 takes values $(1, 1)$ or $(1, -1)$, each with probability $1/2$.

This 2-dimensional walk can be viewed as a space-time diagram of a 1-dimensional simple symmetric random walk:



Interesting combinatorics here, related to the Bohnenblust–Spitzer algorithm; see Steele (2002).

Behaviour of L_n for this case is largely an open problem.

Proof idea: Martingale differences

Let $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. Define $D_{n,i} = \mathbb{E}[L_n - L_n^{(i)} \mid \mathcal{F}_i]$.

Lemma 7

(i) $L_n - \mathbb{E}L_n = \sum_{i=1}^n D_{n,i}$.
(ii) $\mathbb{V}ar(L_n) = \sum_{i=1}^n \mathbb{E}(D_{n,i}^2)$.

Sketch proof.

As Z_i' is independent of $\mathcal{F}_i$, $\mathbb{E}[L_n^{(i)} \mid \mathcal{F}_i] = \mathbb{E}[L_n^{(i)} \mid \mathcal{F}_{i-1}] = \mathbb{E}[L_n \mid \mathcal{F}_{i-1}]$.

So $D_{n,i} = \mathbb{E}[L_n \mid \mathcal{F}_i] - \mathbb{E}[L_n \mid \mathcal{F}_{i-1}]$; a standard construction of a martingale difference sequence.

$$\sum_{i=1}^n D_{n,i} = \mathbb{E}[L_n \mid \mathcal{F}_n] - \mathbb{E}[L_n \mid \mathcal{F}_0] = L_n - \mathbb{E}L_n.$$

Now use orthogonality of martingale differences. □

Aside: Upper bounds

Lemma 3 with the conditional Jensen inequality gives:

$$\mathbb{V}\mathrm{ar}(L_n) \leq \sum_{i=1}^{n} \mathbb{E}[(L_n - L_n^{(i)})^2].$$

A related result, the Efron–Stein inequality, says

$$\mathbb{V}\mathrm{ar}(L_n) \leq \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}[(L_n - L_n^{(i)})^2].$$

It is this latter result that Snyder & Steele used to obtain their upper bound.

Cauchy formula revisited

We need to study $D_{n,i} = \mathbb{E}[L_n - L_n^{(i)} \mid \mathcal{F}_i]$.

We have the Cauchy formula for L_n, and similarly for $L_n^{(i)}$, so that

$$L_n - L_n^{(i)} = \int_0^{\pi} \Delta_{n,i}(\theta) \mathrm{d}\theta,$$

where

$$\Delta_{n,i}(\theta) = \left(M_n(\theta) - M_n^{(i)}(\theta) \right) - \left(m_n(\theta) - m_n^{(i)}(\theta) \right),$$

where, similarly to before,

$$M_n^{(i)}(\theta) = \max_{0 \le j \le n} (S_j^{(i)} \cdot \mathbf{e}_\theta), \quad m_n^{(i)}(\theta) = \min_{0 \le j \le n} (S_j^{(i)} \cdot \mathbf{e}_\theta).$$

Proof idea: Control of extrema

We want to understand the relationship between $M_n(\theta)$, $m_n(\theta)$ and $M_n^{(i)}(\theta)$, $m_n^{(i)}(\theta)$ (resampled versions).

WLOG suppose $\mathbb{E}Z_1 = \mu \mathbf{e}_{\pi/2} = (0, \mu)$, where $\mu > 0$.

Then for each fixed θ, $S_j \cdot \mathbf{e}_\theta$ is a one-dimensional random walk.

Indeed, $S_j \cdot \mathbf{e}_\theta = \sum_{k=1}^{j} Z_k \cdot \mathbf{e}_\theta$, with mean increment $\mathbb{E}[Z_1 \cdot \mathbf{e}_\theta] = \mathbb{E}[Z_1] \cdot \mathbf{e}_\theta = \mu \sin \theta$, which is positive for $\theta \in (0, \pi)$.

So, with high probability, the maximum $M_n(\theta)$ will be achieved nearby step n while the minimum $m_n(\theta)$ will be achieved nearby step 0.

To formalize this needs only the strong law of large numbers, plus some care (need some uniformity in θ).

Proof idea: Control of extrema

To get uniform control, take $\theta \in (\delta, \pi - \delta)$.

Let $\underline{J}_n(\theta) = \arg\min_{0 \le j \le n}(S_j \cdot \mathbf{e}_\theta)$ and $\bar{J}_n(\theta) = \arg\max_{0 \le j \le n}(S_j \cdot \mathbf{e}_\theta)$;

similarly $\underline{J}_n^{(i)}(\theta)$ and $\bar{J}_n^{(i)}(\theta)$ for the walk with Z_i resampled.

Let $E := E_{n,i}(\delta, \gamma)$ be the event that for all $\theta \in (\delta, \pi - \delta)$,

- $\underline{J}_n(\theta) < \gamma n$, $\bar{J}_n(\theta) > (1 - \gamma)n$;
- $\underline{J}_n^{(i)}(\theta) < \gamma n$, $\bar{J}_n^{(i)}(\theta) > (1 - \gamma)n$.

Lemma 8

For any $\delta \in (0, \pi/2)$ *and* $\gamma \in (0, 1)$, $\mathbb{P}(E) \to 1$, *uniformly in* i.

Sketch proof.

Follows from the SLLN. □

Proof idea: Control of extrema

Lemma 9

On E, $\Delta_{n,i}(\theta) = (Z_i - Z_i') \cdot \mathbf{e}_\theta$ for all i with $\gamma n < i < (1-\gamma)n$.

Sketch proof.

On E, both the $\bar{J}$ are $> (1-\gamma)n$ and both the $\underline{J}$ are $< \gamma n$.

It follows that for i in the middle, $\bar{J} = \bar{J}^{(i)}$ and $\underline{J} = \underline{J}^{(i)}$.

So $m_n(\theta) = m_n^{(i)}(\theta)$, and

$$M_n^{(i)}(\theta) = S_{\bar{J}}^{(i)} \cdot \mathbf{e}_\theta = (S_{\bar{J}} - Z_i + Z_i') \cdot \mathbf{e}_\theta = M_n(\theta) + (Z_i' - Z_i) \cdot \mathbf{e}_\theta.$$

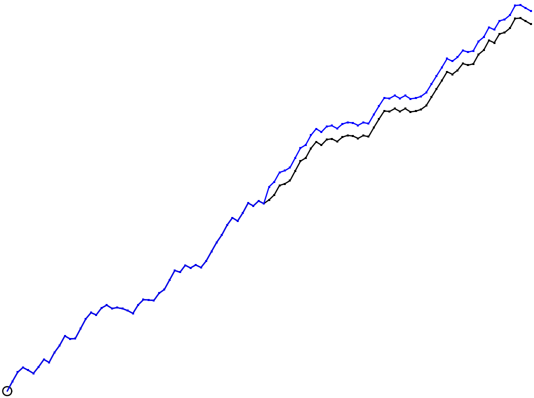
See the picture! □

Proof idea: Control of extrema

So $m_n(\theta) = m_n^{(i)}(\theta)$, and

$$M_n^{(i)}(\theta) = S_{\bar{J}}^{(i)} \cdot \mathbf{e}_\theta = (S_{\bar{J}} - Z_i + Z_i') \cdot \mathbf{e}_\theta = M_n(\theta) + (Z_i' - Z_i) \cdot \mathbf{e}_\theta.$$

See the picture!



Finishing the proofs

The main technical work (details omitted!) now is dealing with the error terms (sending $\delta \to 0$, $\gamma \to 0$, $n \to \infty$).

Up to these error terms, we have shown that

$$D_{n,i} = \mathbb{E}[L_n - L_n^{(i)} \mid \mathcal{F}_i] \approx \int_0^\pi \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i] \mathrm{d}\theta.$$

Here Z_i is $\mathcal{F}_i$-measurable and Z_i' is independent of $\mathcal{F}_i$, so

$$\mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i] = (Z_i - \mathbb{E}Z_1) \cdot \mathbf{e}_\theta.$$

Doing the integral gives

$$D_{n,i} = \mathbb{E}[L_n - L_n^{(i)} \mid \mathcal{F}_i] \approx \frac{2(Z_i - \mathbb{E}Z_1) \cdot \mathbb{E}Z_1}{\|\mathbb{E}Z_1\|}.$$

Finishing the proofs

Formalizing the analysis we get:

Theorem 10

$$n^{-1/2}\left|L_n - \mathbb{E}L_n - \sum_{i=1}^{n}\frac{2(Z_i - \mathbb{E}Z_1)\cdot\mathbb{E}Z_1}{\|\mathbb{E}Z_1\|}\right| \to 0,\ \textit{in } L^2.$$

So, perhaps surprisingly, $L_n - \mathbb{E}L_n$ is well-approximated by a sum of i.i.d. random variables.

Theorems 5 and 6 now follow from Theorem 10 easily enough.

Concluding remarks

The assumption that the Z_i are identically distributed is not essential to the main argument.

For example, let $G_n = \frac{1}{n+1} \sum_{i=0}^{n} S_i = \sum_{i=1}^{n} \frac{n+1-i}{n+1} Z_i$.

$G_0, G_1, \ldots$ is the centre-of-mass process associated with $S_0, S_1, \ldots$.

By convexity, $\text{hull}(G_0, \ldots, G_n) \subseteq \text{hull}(S_0, \ldots, S_n)$.

If $L_n^\star$ is the perimeter length of $\text{hull}(G_0, \ldots, G_n)$, then the statement of Theorem 10 applies to $L_n^\star$ in place of L_n with $\frac{n+1-i}{n+1} Z_i$ in place of Z_i.

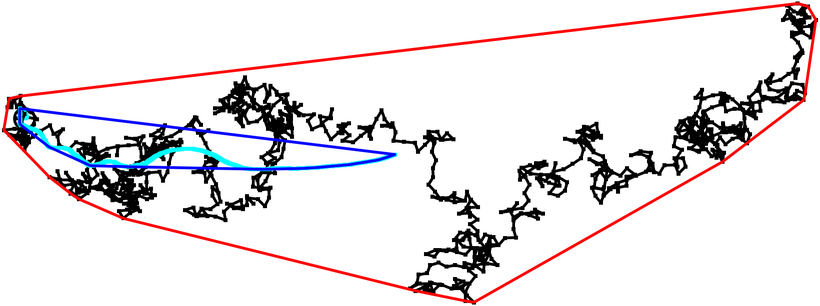
In particular, the analogue of Theorem 5 says that

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{V}\text{ar}(L_n^\star) = s^2/3,$$

where s^2 is the same as before.

Concluding remarks

A picture:



Concluding remarks

Ongoing work: look at A_n, the area of hull$(S_0, \ldots, S_n)$.

There's a (more complicated) formula for $\mathbb{E}(A_n)$, due to Barndorff-Nielsen and Baxter (1963).

We can show

- $\mathbb{V}\text{ar}(A_n) = O(n^3)$ in the case with drift;
- $\mathbb{V}\text{ar}(A_n) = O(n^2)$ in the zero-drift case.

We expect these bounds are of the correct order.

There's a (more complicated) Cauchy formula here, too, but to get a precise limit statement (or even a lower bound) in this case looks harder.

Several interesting open problems. . .

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