Tail behaviour of stationary distribution for Markov chains with asymptotically zero drift

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April, 2014

Outline

Object of study

One-dimensional homogenous Markov chain on R^+.

$$X_n, n = 0, 1, 2, \ldots$$

Let $\xi(x)$ be a random variable corresponding to a jump at point x, i.e.

$$\mathbf{P}(\xi(x) \in B) = \mathbf{P}(X_{n+1} - X_n \in B \mid X_n = x).$$

Let

$$m_k(x) := \mathbf{E}[\xi(x)^k].$$

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Main assumptions

- *Small drift:*

$$m_1(x) \sim \frac{-\mu}{x}, \quad x \to \infty;$$

- *Finite variance:*

$$m_2(x) \to b, \quad x \to \infty.$$

Questions

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then X_n is ergodic (Lamperti)
What can one say about the stationary distribution?
2. If
$$2xm_1(x) - m_2(x) \ge \varepsilon$$
then X_n is transient .
What can one say about the renewal (Green) function
$$H(x) = \sum_{n=0}^{\infty} \mathbf{P}(X_n \le x), \quad x \to \infty?$$

Continuous time - Bessel-like diffusions

Let X_t be the solution to SDE

$$dX_t = \frac{-\mu(X_t)}{X_t} dt + \sigma(X_t) dW_t, \quad X_0 = x > 0,$$

where $\mu(x) \to \mu$ and $\sigma(x) \to \sigma$.
For Bessel processes $\mu(x) = const$ and $\sigma(x) = const$.

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We can use forward Kolmogorov equations to find exact stationary density

$$0 = \frac{d}{dx}\left(\frac{\mu(x)}{x} p(x)\right) + \frac{1}{2}\frac{d^2}{dx^2}\left(\sigma^2(x) p(x)\right)$$

to obtain

$$p(x) = \frac{2c}{\sigma^2(x)} \exp\left\{-\int_0^x \frac{2\mu(y)}{y\sigma^2(y)} dy\right\}, \qquad c > 0.$$

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Then,

$$p(x) \approx C \exp\left\{-\int_1^x \frac{2\mu}{b}\frac{dy}{y}\right\} \sim Cx^{-2\mu/b}$$

and

$$\pi(x, +\infty) = \int_x^\infty p(y) dy \approx Cx^{-2\mu/b+1}$$

Simple Markov chain

Markov chain on $\mathbf{Z}$

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with solution

$$\pi(x) = \pi(0) \prod_{k=1}^{x} \frac{p_+(k-1)}{p_-(k)} = \pi(0) \exp\left\{ \sum_{k=1}^{x} (\log p_+(k-1) - \log p_-(k)) \right\},$$

Asymptotics for the tail of the stationary measure

Theorem

Suppose that, as $x \to \infty$,

$$m_1(x) \sim -\frac{\mu}{x}, \quad m_2(x) \to b \quad \text{and} \quad 2\mu > b. \tag{1}$$

Suppose some technical conditions and

$$m_3(x) \to m_3 \in (-\infty, \infty) \quad \text{as } x \to \infty \tag{2}$$

and, for some $A < \infty$,

$$\mathbf{E}\{\xi^{2\mu/b+3+\delta}(x); \xi(x) > Ax\} \;=\; O(x^{2\mu/b}). \tag{3}$$

Then there exist a constant $c > 0$ *such that*

$$\pi(x, \infty) \sim cxe^{-\int_0^x r(y)dy} = cx^{-2\mu/b+1}\ell(x) \quad \text{as } x \to \infty.$$

Known results

- Menshikov and Popov (1995) investigated Markov Chains on $\mathbb{Z}^+$ with bounded jumps and showed that

$$c_- x^{-2\mu/b-\varepsilon} \leq \pi(\{x\}) \leq c_+ x^{-2\mu/b+\varepsilon}.$$

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- Korshunov (2011) obtained the following estimate

$$\pi(x,\infty) \leq c(\varepsilon) x^{-2\mu/b+1+\varepsilon}.$$

General approach - random walk example

Consider a classical example, of Lindley recursion

$$W_{n+1} = (W_n + \xi_n)^+, n = 0, 2, \ldots, W_0 = 0,$$

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This is an ergodic Markov chain (Note drift is not small).
A classical approach consists of three key steps
Step 1: Reverse time and consider a random walk

$$S_n = \xi_1 + \cdots + \xi_n, n = 1, 2 \ldots, S_0 = 0.$$

Then,

$$W_n \xrightarrow{d} W = \sup_{n \geq 0} S_n.$$

Random walks ctd.

Step 2: Exponential change of measure.
Assuming that there exists $\varkappa > 0$ such that

$$\mathbf{E}[e^{\varkappa\xi}] = 1, \quad \mathbf{E}[\xi e^{\varkappa\xi}] < \infty 1$$

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one can perform change of measure

$$\mathbf{P}(\widehat{\xi}_n \in dx) = e^{\varkappa x}\mathbf{P}(\xi_n \in dx), \quad n = 1, 2, \ldots$$

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Under new measure $\widehat{S}_n = \widehat{\xi}_1 + \cdots + \widehat{\xi}_n$ and $\mathbf{E}\widehat{\xi}^1 > 0$

$$\widehat{S}_n \to +\infty, \quad \text{and} \quad S_n \to -\infty.$$

Random walks ctd.

Step 3: Use renewal theorem for $\widehat{S}_n$.

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This step uses ladder heights construction and represents

$$\mathbf{P}(M \in dx) = CH(dx) = Ce^{-\varkappa x}\widehat{H}(dx).$$

Now one can apply standard renewal theorem to $\widehat{H}(dy) \sim dy/c$ to obtain

$$\mathbf{P}(M \in dx) \sim ce^{-\varkappa x}, \quad x \to \infty.$$

Asymptotically homogeneous Markov chains

One can repeat this programme for asymptotically homogenous Markov chains. Namely, assume

$$\xi(x) \xrightarrow{d} \xi, \quad x \to \infty,$$

where

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Borovkov and Korshunov (1996) showed that if

$$\sup_x \mathbf{E} e^{\varkappa\xi(x)} < \infty, \int_0^\infty \left(\int_{\mathbf{R}} e^{\varkappa t} |\mathbf{P}(\xi(x) < t) - \mathbf{P}(\xi < t)| dt \right) dx$$

then

$$\pi(x, \infty) \sim C e^{-\varkappa x}, \quad x \to \infty, \quad x \to \infty.$$

Problems in our case

Problem 1 (easier) In our case drift

$$\mathbf{E}\xi(x) \to 0, \quad x \to \infty.$$

Hence, for

$$1 = \mathbf{E}\exp\{\varkappa\xi(x)\} \approx 1 + \varkappa\mathbf{E}\xi(x), \quad x \to \infty$$

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Hence, exponential change of measure does not work.

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Problem 2 Suppose we managed to make a change of measure. As a result

$$\widehat{X}_n \stackrel{a.s}{\to} +\infty$$

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Then, there is no renewal theorem about

$$\widehat{H}(x) = \sum_{n=0}^{\infty} \mathbf{P}(X_n \le x).$$

Main reason for that

$$\frac{\widehat{X}_n}{n^c} \xrightarrow{d} Gamma(\alpha, \beta)$$

which makes the problem difficult.

Harmonic function

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Let B be a Borel set in $\mathbf{R}^+$ with $\pi(B) > 0$, in our case $B = [0, x_0]$. Let

$$\tau_B := \min\{n \geq 1 : X_n \in B\}.$$

Note $\mathbf{E}_x \tau_B < \infty$ for every x.
$V(x)$ is a harmonic function for X_n killed at the time of the first visit to B, if

$$V(x) = \mathbf{E}_x\{V(X_1); \tau_B > 1\} = \mathbf{E}_x\{V(X_1); X_1 \notin B\}$$

If V is harmonic then

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1. It is not clear that such a (positive) function $V(x)$ exists
2. Some estimates on $V(x)$ are required for further analysis.

Construction of the harmonic function

We start with a harmonic function (scale function) for the corresponding diffusion.

$$dX_t = \frac{-\mu(X_t)}{X_t}dt + \sigma(X_t)dW_t, \quad X_0 = x > 0.$$

For the diffusion this function solves

$$0 = \frac{-\mu(x)}{x}\frac{d}{dx}U(x) + \frac{\sigma(x)^2}{2}\frac{d^2}{dx^2}U(x), \quad x \notin B$$
$$0 = U(x), \quad x \in B = [0, x_0].$$

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Namely the corresponding stopped process $X_{t \wedge \tau_B}$ is a martingale.
The solution is given by

$$U(x) := \int_{x_0}^{x} e^{R(y)} dy \;\text{ for } x \geq x_0, \text{ where } R(y) = \int_{x_0}^{y} r(z) dz, \; r(z) = \frac{2\mu(z)}{\sigma^2(z)}.$$

Construction of the harmonic function ctd.

Note that

$$r(z) = \frac{2\mu}{b}\frac{1}{z} + \frac{\varepsilon(z)}{z}.$$

Hence

$$U(x) \sim x^{2\mu/b+1} l(x), \quad x \to \infty, l(x) - \text{slowly varying.}$$

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U is *not* harmonic for the initial Markov chain X_n. However if the correction

$$u(x) = \mathbf{E} U(X_1) - u(x), \quad \text{is small } x \to \infty,$$

then

$$V(x) := U(x) + \mathbf{E}_x \sum_{n=0}^{\tau_B - 1} u(X_n)$$

is well-defined, non-negative and harmonic for X_n .

Construction of the harmonic function ctd.

Function $u(x)$ by the Taylor expansion

$$\begin{aligned} u(x) &= \mathbf{E}U(X_1) - u(x) \\ &= U'(x)\mathbf{E}_X(X_1 - x) + \frac{1}{2}U''(x)\mathbf{E}_X(X_1 - x)^2 \\ &\quad + \frac{1}{6}U'''(x + \theta(x))\mathbf{E}_X(X_1 - x)^3. \end{aligned}$$

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Now the first 2 terms disappear since $U(x)$ is harmonic for the diffusion. Hence,

$$u(x) \sim CU'''(x) \sim C\frac{U(x)}{x^3}.$$

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This is sufficient to ensure the finiteness

$$\mathbf{E}_x \sum_{n=0}^{\tau_B - 1} |u(X_n)| < \infty.$$

Change of measure

As V is well-defined we can perform the change of measure (Doob's h-transform).
Let $\widehat{X}_n$ be a Markov Chain with the following transition kernel

$$\mathbf{P}_z\{\widehat{X}_1 \in dy\} = \frac{V(y)}{V(z)}\mathbf{P}_z\{X_1 \in dy; \tau_B > 1\}$$

Since V is harmonic, then we also have

$$\mathbf{P}_z\{\widehat{X}_n \in dy\} = \frac{V(y)}{V(z)}\mathbf{P}_z\{X_n \in dy; \tau_B > n\} \quad \text{for all } n.$$

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Note $V(x) \sim U(x) \sim x^{2\mu/b+1}$. Hence, this change of measure is non-exponential.

Change of measure for stationary distribution

Balance equation for π

$$\pi(dy) = \int_B \pi(dz) \sum_{n=0}^{\infty} \mathbf{P}_z\{X_n \in dy; \tau_B > n\}.$$

Changing the measure

$$\begin{aligned}
\pi(dy) &= \frac{1}{V(y)} \int_B \pi(dz) V(z) \sum_{n=0}^{\infty} \mathbf{P}_z\{\widehat{X}_n \in dy\} \\
&= \frac{\widehat{H}(dy)}{V(y)} \int_B \pi(dz) V(z),
\end{aligned}$$

where $\widehat{H}$ is the renewal measure generated by the chain $\widehat{X}_n$ with initial distribution

$$\mathbf{P}\{\widehat{X}_0 \in dz\} = \widehat{c}\pi(dz)V(z), \ z \in B \quad \text{and } \widehat{c} := \Big(\int_B \pi(dz)V(z)\Big)^{-1}.$$

Renewal theorem

Therefore,

$$\begin{aligned} \pi(x,\infty) &= \widehat{c}\int_x^\infty \frac{1}{V(y)}d\widehat{H}(y) \\ &\sim \widehat{c}\int_x^\infty \frac{1}{U(y)}d\widehat{H}(y) \quad \text{as } x\to\infty, \end{aligned}$$

as $V(x)\sim U(x)$.

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as $V(x) \sim U(x)$.
We are facing second problem now - what is the asymptotics for

$$\widehat{H}(x) = \sum_{n=1}^{\infty} \mathbf{P}(\widehat{X}_n \le x), \quad x \to \infty.$$

Renewal theorem

First, the change of measure gives $\widehat{X}_n$ of the same type, but now transient

- *Small drift:*

$$\widehat{m}(x) = \mathbf{E}\left[\widehat{X}_1 - \widehat{X}_0 \;\middle|\; \widehat{X}_0 = x\right] \sim \frac{\mu}{x}, \quad x \to \infty;$$

- *Finite variance:*

$$\widehat{\sigma}^2(x) = \mathbf{E}\left[(\widehat{X}_1 - \widehat{X}_0)^2 \;\middle|\; \widehat{X}_0 = x\right] \to b, \quad x \to \infty.$$

Lower bound for the renewal theorem

Lower bound follows from weak convergence

$$\frac{\widehat{X}_n^2}{n} \xrightarrow{d} \Gamma$$

with mean $2\mu + b$ and variance $(2\mu + b)2b$.

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with mean $2\mu + b$ and variance $(2\mu + b)2b$.
Then,

$$\begin{aligned} \widehat{H}(x) &\geq \sum_{n=0}^{[Bx^2]} \mathbf{P}_y\{X_n \leq x\} \\ &= \sum_{n=0}^{[Bx^2]} (\Gamma(x^2/n) + o(1)) \\ &= x^2 \int_0^B \Gamma(1/z)dz + o(x^2). \end{aligned}$$

and

$$\int_0^B \Gamma(1/z)dz \to \frac{1}{2\mu - b} \quad \text{as } B \to \infty,$$

we conclude the lower bound

Renewal theorem (Asymptotics for the Green function)

Theorem

Consider a transient Markov chain X_n. If $m(x) \sim \mu/x$ and $\sigma^2(x) \to b > 0$ as $x \to \infty$, and $2\mu > b$, then, for any initial distribution of the chain X,

$$H(x) \sim \frac{x^2}{2\mu - b} \quad \text{as } x \to \infty,$$

where $H(x) = \sum_{n=0}^{\infty} \mathbf{P}(X_n \leq x)$.

Stationary measure

We can continue with stationary measure

$$\begin{aligned}\pi(x,\infty) &\sim \widehat{c}\int_x^\infty \frac{1}{U(y)}d\widehat{H}(y)\\ &\sim \widehat{c}\int_x^\infty \frac{1}{y^{2\mu/b+1}}l(y)d\frac{y^2}{(2\mu-b)}\\ &\sim 2\frac{\widehat{c}}{2\mu-b}\int_x^\infty \frac{1}{y^{2\mu/b+1}}l(y)dy\\ &\sim \frac{C}{x^{2\mu/b+1}}l(x).\end{aligned}$$

To apply integral renewal theorem we integrate by parts

Harmonic functions vs Lyapunov functions

- Lyapunov functions We choose an explicit function x^a, e^{hx}. Therefore, there are no problems with regularity properties. One can use Taylor expansion to obtain a submartingale ot supermartingale and hence bounds.
- Harmonic functions Explicit expressions are rarely known. Special properties should be derived. Harmonic functions lead to martingales and more accurate estimates.

Further developments

We plan to consider a problem with the following decay of the drift

$$m(x) = \mathbf{E}\left[X_1 - X_0 \;\mid\; X_0 = x\right] \sim \frac{-\mu}{x^a}, \quad x \to \infty,$$

where $a \in (0,1)$.
One can expect the following decay

$$\pi(x, +\infty) \sim \exp\{-x^{1-a}\}, \quad x \to \infty..$$

References

Main References

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