# Non-homogeneous random walks

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# Introduction

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Consider $X_t$, $t \in \mathbb{Z}_+$ a nearest-neighbour random walk on $\mathbb{Z}_+$.

We are interested in random quantities such as

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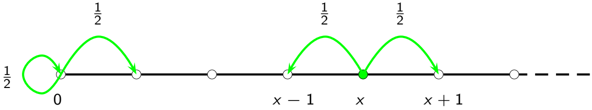
- etc. . .

describing the process $(X_t)_{t \ge 0}$ at **large but finite** times.

# Introduction (cont.)

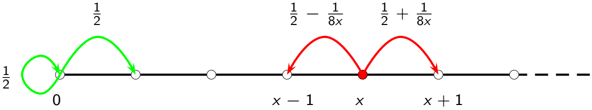
How do these quantities behave (tails, asymptotics, . . . ) for this random walk?:




Symmetric (**zero drift**) walk with reflection at the origin.

# Introduction (cont.)

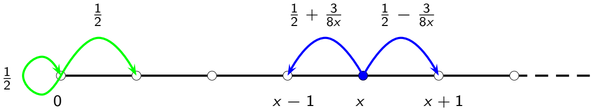
What about this random walk?:




Non-homogeneous random walk with **asymptotically zero drift** $\frac{1}{4x}$.

# Introduction (cont.)

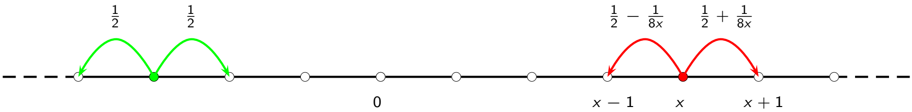
Or this one?:




Another walk with asymptotically zero drift $-\frac{3}{4x}$.

# Introduction (cont.)

Or this combination?:




Symmetric walk for non-positive sites, non-homogeneous walk with **asymptotically zero drift** $\frac{1}{4x}$ for positive sites.

# Introduction (cont.)

I will describe answers to these questions. I will emphasize that the answers depend **not at all** on the nearest-neighbour structure, bounded jumps, or even the Markov property.

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All that really matters are the **first two moment functions** of the increments, i.e.,

$$\mathrm{E}[X_{t+1} - X_t \mid X_t = x] \quad \text{and} \quad \mathrm{E}[(X_{t+1} - X_t)^2 \mid X_t = x]$$

and some **regenerative** structure for the process (so excursions are well defined).

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First I will give a general overview of **non-homogeneous random walks**.

# Random walk origin

- Lord Rayleigh's theory of sound (1880s)
- Louis Bachelier's thesis on random models of stock prices (1900)
- Karl Pearson's theory of random migration (1905-06)
- Einstein's theory of Brownian motion (1905-08)

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# Simple random walk

Let $X_t$ be symmetric simple random walk (SRW) on $\mathbb{Z}^d$, i.e., given $X_1, \ldots, X_t$, the new location $X_{t+1}$ is uniformly distributed on the $2d$ adjacent lattice sites to $X_t$.

**Theorem (Pólya 1921)**

*SRW is recurrent if $d = 1$ or $d = 2$, but transient if $d \geq 3$.*

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**Theorem (Pólya 1921)**

*SRW is recurrent if $d = 1$ or $d = 2$, but transient if $d \geq 3$.*

"A drunk man will find his way home, but a drunk bird may get lost forever." —Shizuo Kakutani

# Lyapunov functions

- There are several proofs of Pólya's theorem available, typically using combinatorics or electrical network theory.
- These classical approaches are of limited use if one starts to generalize or perturb the model slightly.
- Lamperti (1960) gave a very robust approach, based on the method of **Lyapunov functions**.
- Reduce the $d$-dimensional problem to a 1-dimensional one by taking $Z_t := \|X_t\|$.
- $Z_t = 0$ if and only if $X_t = 0$, but the reduction of dimensionality comes at a (modest) price: $Z_t$ is **not** in general a Markov process.

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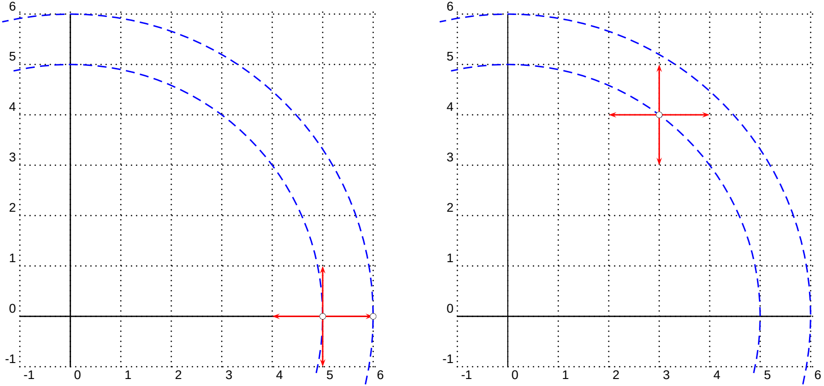
# Lyapunov functions (cont.)

E.g. in $d = 2$, consider the two events $\{X_t = (3,4)\}$ and $\{X_t = (5,0)\}$. Both imply $Z_t = 5$, but in only one case there is positive probability of $Z_{t+1} = 6$.




So our methods cannot rely on the Markov property.

# Lyapunov functions (cont.)

- Elementary calculations based on Taylor's theorem and properties of the increments $\Delta_n = X_{n+1} - X_n$ show that

$$\mathrm{E}\left[Z_{t+1} - Z_t \mid X_1, \ldots, X_t\right] = \frac{1}{2Z_t}\left(1 - \frac{1}{d}\right) + O(Z_t^{-2}),$$

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- In particular, $Z_t$ is a stochastic process on $[0, \infty)$ with **asymptotically zero drift**.
- Loosely speaking, if

$$\mu_k(z) = \mathrm{E}\big[(Z_{t+1} - Z_t)^k \mid Z_t = z\big],$$

we have $\mu_1(z) \sim \frac{1}{2z}\left(1 - \frac{1}{d}\right)$ and $\mu_2(z) \sim \frac{1}{d}$.

# Lamperti's problem

In the early 1960s, Lamperti studied in detail how the asymptotics of a stochastic process $Z_t \in [0, \infty)$ are determined by the first two moment functions of its increments, $\mu_1$ and $\mu_2$.

**Theorem (Lamperti 1960–63)**

*Under mild regularity conditions, the following recurrence classification holds.*

- *If* $2z\mu_1(z) - \mu_2(z) > \varepsilon > 0$, $Z_t$ *is transient.*
- *If* $2z\mu_1(z) + \mu_2(z) < -\varepsilon < 0$, $Z_t$ *is positive-recurrent.*
- *If* $|2z\mu_1(z)| \leq \mu_2(z)$, $Z_t$ *is null-recurrent.*

# Lamperti's problem (cont.)

- In particular, for $Z_t = \|X_t\|$ the norm of SRW,

$$2z\mu_1(z) \sim 1 - \frac{1}{d}, \qquad \text{and} \qquad \mu_2(z) \sim \frac{1}{d}.$$

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- So Pólya's theorem follows.
- This approach allows one to study much more general random walk models, including spatially **non-homogeneous** random walks, and non-Markovian processes.
- More generally, many **near-critical** stochastic systems, if a suitable Lyapunov function exists, can be analysed using Lamperti's theorem.

# Conditions for recurrence?

Consider the more general **non-homogeneous** situation where $X_t$ is a Markov chain on $\mathbb{R}^d$ whose jump distribution may change from place to place.

So now

$$\mu(\mathbf{x}) = \mathsf{E}[X_{t+1} - X_t \mid X_t = \mathbf{x}]$$

is allowed to depend on $\mathbf{x} \in \mathbb{R}^d$.

**Question:** In the non-homogeneous case, is $\mu(\mathbf{x}) = \mathbf{0}$ sufficient for recurrence in $d = 2$?

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**Question:** In the non-homogeneous case, is $\mu(\mathbf{x}) = \mathbf{0}$ sufficient for recurrence in $d = 2$?

**Answer:** No.

**Theorem**

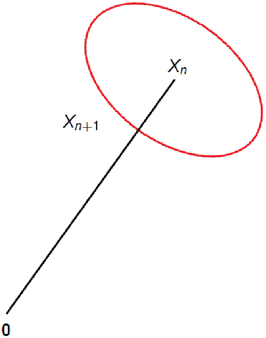
*Let $X_t$ be a non-homogeneous random walk with zero drift, i.e., $\mu(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^d$. There exist such walks that are*

- *transient in $d = 2$;*
- *recurrent in $d \geq 3$.*

# Elliptical random walk

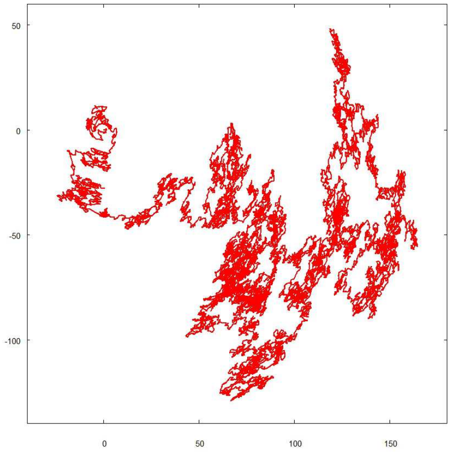
Here is an example of the previous theorem in $d = 2$.

Given $X_t$, suppose that $X_{t+1}$ is distributed (uniformly with respect to the standard parametrization) on an **ellipse** centred at $X_t$ and aligned so that the **minor axis** is in the direction of the vector $X_t$.




This zero-drift non-homogeneous random walk in $\mathbb{R}^2$ is **transient**.

# Elliptical random walk



# Asymptotically zero drift

Lamperti published a series of pioneering papers in the early 1960s investigating the asymptotically zero drift regime ($\mu(\mathbf{x}) \to \mathbf{0}$ as $\|\mathbf{x}\| \to \infty$) which is the natural setting in which to probe the recurrence-transience transition.

A zero drift non-homogeneous random walk on $\mathbb{R}^d$ can always be made recurrent or transient (whichever is desired) by an **asymptotically small perturbation** of the drift field.

More precisely, changing the drift $\mu(\mathbf{x})$ by $O(\|\mathbf{x}\|^{-1})$ is sufficient to achieve this.

Now we return to the **one-dimensional setting** to address the specific questions posed in the introduction.

# One-dimensional case

For simplicity of presentation, we take $X_t$ to be **Markov** (time-homogeneous and irreducible) and its state space $\mathcal{S} \subseteq [0, \infty)$ to be **locally finite** with $0 \in \mathcal{S}$.

The Markov assumption is not necessary, but we do need a regenerative structure.

We assume the following moment conditions on the increments $\Delta_t := X_{t+1} - X_t$: for some $c \in \mathbb{R}$ and $s^2 \in (0, \infty)$,

$$\mathrm{E}\left[\Delta_t \mid X_t = x\right] \approx \frac{c}{x}, \qquad \mathrm{E}\left[\Delta_t^2 \mid X_t = x\right] \approx s^2,$$

where '$\approx$' means that we are ignoring some higher order terms as $x \to \infty$.

# Recurrence classification

Let $c$ and $s^2$ be defined as above,

$$\mathsf{E}\big[\Delta_t \mid X_t = x\big] \approx \frac{c}{x}\,, \qquad \mathsf{E}\big[\Delta_t^2 \mid X_t = x\big] \approx s^2\,.$$

The key quantity turns out to be

$$r := -\frac{2c}{s^2} \in \mathbb{R}\,.$$

### Theorem (Lamperti)

*Under mild conditions, $X_t$ is*

- *transient if $r < -1$,*
- *null-recurrent if $-1 \le r \le 1$,*
- *positive-recurrent if $r > 1$.*

# Excursions

For the rest of this talk we focus on the **recurrent** case $r > -1$, and examine in detail the **excursion** structure of the process.

Start the process from $X_0 = 0$ and consider

$$\tau := \min\{t > 0 : X_t = 0\}\,.$$

We study **path properties** of $X_0, X_1, \dots, X_t$ as $t \to \infty$ via a study of the **excursions** $X_0, X_1, \dots, X_\tau$.

# Excursion maxima

To illustrate our approach, we first consider

$$M := \max_{0 \le t \le \tau} X_t,$$

the maximum attained by the walk over an excursion.

Consider the **Lyapunov function** $Y_t := X_t^\gamma$, $\gamma > 0$.

A Taylor's formula calculation shows that

$$\begin{aligned} Y_{t+1} - Y_t = (X_t + \Delta_t)^\gamma - X_t^\gamma = X_t^\gamma \left[ \left( 1 + \frac{\Delta_t}{X_t} \right)^\gamma - 1 \right] \\ \approx \gamma \Delta_t X_t^{\gamma - 1} + \frac{\gamma(\gamma - 1)}{2} \Delta_t^2 X_t^{\gamma - 2}, \end{aligned}$$

under suitable conditions (e.g. a $2 + \varepsilon$ moment bound on $\Delta_t$).

# Excursion maxima (cont.)

As a result,

$$\begin{aligned} \mathrm{E}\left[Y_{t+1} - Y_t \mid X_t = x\right] &\approx \gamma \frac{c}{x} x^{\gamma-1} + \frac{\gamma(\gamma-1)}{2} s^2 x^{\gamma-2} \\ &= \frac{\gamma}{2} x^{\gamma-2} \left(2c + (\gamma-1)s^2\right) . \end{aligned}$$

The last expression is 0 if $\gamma = 1 - \frac{2c}{s^2} = 1 + r$.

In other words, for $\gamma = 1 + r$, $X_t^\gamma$ is almost a martingale. A small perturbation in either direction will give a submartingale or a supermartingale.

Then optional stopping ideas give

$$\Pr\left[X_t \text{ hits } x \text{ before returning to } 0\right] \approx x^{-1-r} .$$

# Excursion maxima (cont.)

The relation

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implies

$$\Pr\big[M > x\big] \approx x^{-1-r}.$$

So $E[M^p] < \infty$ if and only if $p < 1 + r$.

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So $\mathrm{E}[M^p] < \infty$ if and only if $p < 1 + r$.

**For example**: In the zero drift case, $\Pr[M > x] \approx 1/x$ and $\mathrm{E}[M] = \infty$.

# Excursion duration

On the event that $X_t$ reaches large $x$ during the excursion, semimartingale estimates can be used to show that with good probability, the walk spends time of order $x^2$ before it returns to 0.

So $\Pr[\tau > x^2] \approx \Pr[M > x] \approx x^{-1-r}$.

That is, $\Pr[\tau > x] \approx x^{-\frac{1+r}{2}}$.

(Actually this sketched argument only gives a lower bound. The upper bound uses semimartingale ideas of Aspandiiarov, Iasnogorodskii and Menshikov.)

# Number of excursions

The duration of an excursion has tail $\Pr[\tau > x] \approx x^{-\frac{1+r}{2}}$.

E.g. for the zero-drift case, this exponent is $1/2$.

Let $N(t)$ be the **number of excursions** (i.e., the number of visits to 0) by time $t$.

An inversion of the law of large numbers shows that:

- If $-1 < r \leq 1$ (the null-recurrent case), then

$$N(t) \approx t^{\frac{1+r}{2}} \quad \text{a.s.}$$

- If $r > 1$ (the **ergodic case**), then

$$t^{-1} N(t) \to \mathsf{E}[\tau]^{-1} \quad \text{a.s.},$$

  which is a constant.

# Running maximum

We have

$$\max_{0 \le s \le t} X_s \approx \max \text{ of } N(t) \text{ copies of } M\,.$$

The tail bounds on $M$ then give

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There are 2 cases:

- If $-1 < r \le 1$ (**null-recurrent** case), then

$$\max_{0 \le s \le t} X_s \approx t^{\frac{1}{2}} \,.$$

- If $r > 1$ (**ergodic** case), then

$$\max_{0 \le s \le t} X_s \approx t^{\frac{1}{1+r}} \,.$$

# Excursion sums

Now we are going to work towards an understanding of the **path integrals**

$$S_t^{(\alpha)} := \sum_{s=0}^{t} X_s^{\alpha}, \quad \alpha > 0.$$

Our particular motivation was initially to understand the behaviour of the **centre of mass** $G_t := \frac{1}{1+t} S_t^{(1)}$.

Again a first step is to examine a single excursion. Set

$$\xi^{(\alpha)} := \sum_{s=0}^{\tau-1} X_s^{\alpha}.$$

# Excursion sums (cont.)

We use a similar argument to before. With probability about $x^{-1-r}$, the walk reaches $x$ during the excursion.

On this event, with good probability, the walk then spends time of order $x^2$ at distance at least $x/2$, say.

This accumulates an excursion sum of order $x^2 \cdot x^\alpha$.

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This accumulates an excursion sum of order $x^2 \cdot x^\alpha$.

So $\Pr\left[\xi^{(\alpha)} > x^{2+\alpha}\right] \approx x^{-1-r}$. In other words,

$$\Pr\left[\xi^{(\alpha)} > x\right] \approx x^{-\frac{1+r}{2+\alpha}} .$$

In particular, $E[\xi^{(\alpha)}] < \infty$ if and only if $r > 1 + \alpha$.

# Path integrals

Again, the argument sketched gives the lower bound. The upper bound is straightforward from the fact that

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Now

$$S_t^{(\alpha)} \approx \sum \text{ of } N(t) \text{ copies of } \xi^{(\alpha)} .$$

The tail bounds for $\xi^{(\alpha)}$ then give:

- If $r \leq 1 + \alpha$, then
$$S_t^{(\alpha)} \approx N(t)^{\frac{1+r}{2+\alpha}} .$$
- If $r > 1 + \alpha$, then
$$S_t^{(\alpha)} \approx N(t) .$$

# Path integrals (cont.)

Combining this with our results for $N(t)$ gives the following 3 cases:

- If $-1 < r \leq 1$ (null-recurrent case) then

$$S_t^{(\alpha)} \approx t^{\frac{2+\alpha}{2}}\,.$$

- If $1 < r \leq 1+\alpha$ (weakly ergodic case) then

$$S_t^{(\alpha)} \approx t^{\frac{2+\alpha}{1+r}}\,.$$

- If $r > 1+\alpha$ (strongly ergodic case) then

$$t^{-1} S_t^{(\alpha)} \to \nu_\alpha \in (0,\infty)\,.$$

# Centre of mass process

As a corollary, we obtain the following results for the **centre of mass process** $G_t$.

- If $-1 < r \leq 1$ then $G_t \approx t^{\frac{1}{2}}$.
- If $1 < r \leq 2$ then $G_t \approx t^{\frac{2-r}{1+r}}$.
- If $r > 2$ then $G_t \to \nu_1$.

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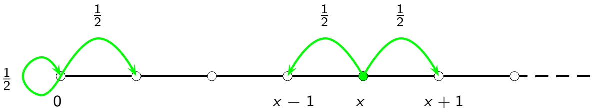
Comparing the exponents for $G_t$ to those of the **maximum process** $\max_{0 \leq s \leq t} X_s$:

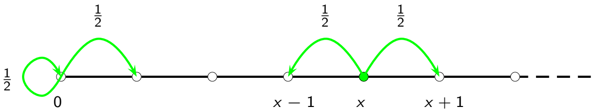
- They coincide (taking value $\frac{1}{2}$) in the **null-recurrent** case.
- In the **positive-recurrent** case, $\frac{1}{1+r} > \frac{2-r}{1+r}$ for $r > 1$. The intuition here is that in the positive-recurrent case, the process rarely visits the scale of the maximum, so $G_t \ll \max_{0 \leq s \leq t} X_s$.

# Some simple examples




Symmetric (**zero drift**) walk with reflection at the origin.

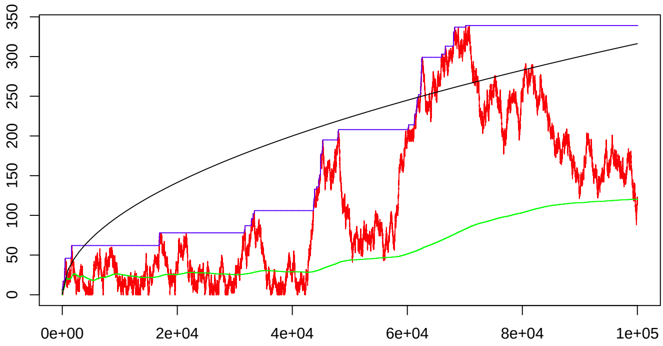
# Some simple examples




Symmetric (**zero drift**) walk with reflection at the origin. Here

- $\Pr[M > x] \approx x^{-1}$.
- $\Pr[\tau > x] \approx x^{-\frac{1}{2}}$.
- $\Pr[\xi^{(\alpha)} > x] \approx x^{-\frac{1}{2+\alpha}}$.
- $\max_{0 \le s \le t} X_s \approx t^{\frac{1}{2}}$.
- $G_t \approx t^{\frac{1}{2}}$.

# Some simple examples (cont.)

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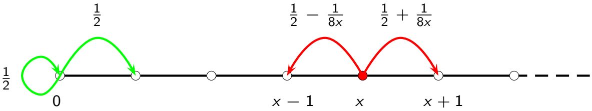
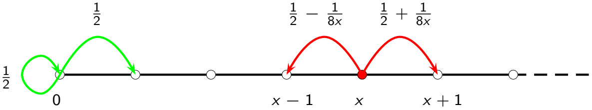



Non-homogeneous random walk with **asymptotically zero drift** $\frac{1}{4x}$ and $r = -\frac{2c}{s^2} = -\frac{1}{2}$, so **null-recurrent**.

# Some simple examples (cont.)

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- $\Pr[M > x] \approx x^{-\frac{1}{2}}$.
- $\Pr[\tau > x] \approx x^{-\frac{1}{4}}$.
- $\Pr[\xi^{(\alpha)} > x] \approx x^{-\frac{1}{4+2\alpha}}$.
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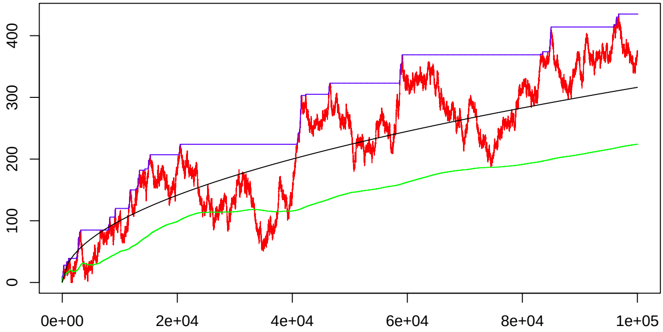
# Some simple examples (cont.)

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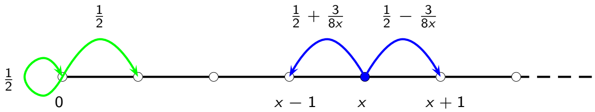


Non-homogeneous random walk with asymptotically zero drift $-\frac{3}{4x}$ and $r = -\frac{2c}{s^2} = \frac{3}{2}$ so **positive-recurrent**.

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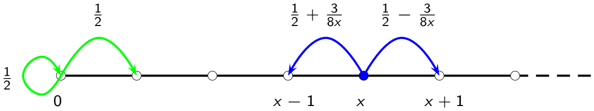


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- $\Pr[M > x] \approx x^{-\frac{5}{2}}$.
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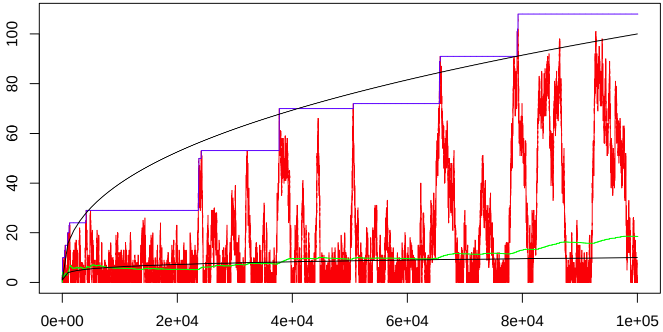


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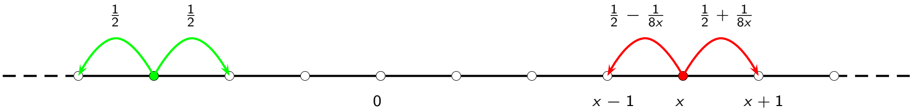
# Further illustration: A walk on $\mathbb{Z}$




Consider a nearest-neighbour random walk on $\mathbb{Z}$.

- From $x \le 0$, the walk takes symmetric jumps ($\pm 1$ with probability $\frac{1}{2}$ each).
- From $x > 0$, the walk jumps to $x \pm 1$ with probabilities $\frac{1}{2} \pm \frac{1}{8x}$.

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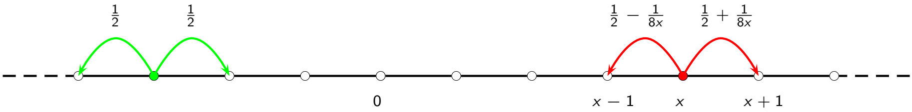


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Restricting the process to either half-line gives a **null-recurrent** process with **diffusive** ($t^{1/2}$) scaling.

What about the combined process?

# Illustration: A walk on $\mathbb{Z}$ (cont.)

In fact, there is a separation of scales:

$$\max_{0 \le s \le t} X_s \approx t^{1/2}\,, \qquad \min_{0 \le s \le t} X_s \approx -t^{1/4}\,.$$

Moreover, $G_t \approx t^{1/2}$ (positive!).

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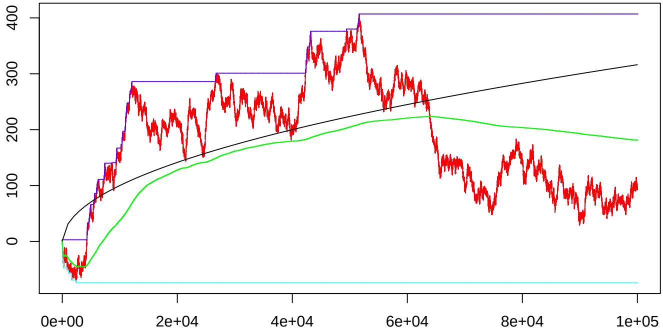
Moreover, $G_t \approx t^{1/2}$ (positive!).

The intuition here is that the walk makes a comparable number of positive and negative excursions, but the positive ones have heavier-tailed durations, and so occupy a dominant fraction of the time.

# Illustration: A walk on $\mathbb{Z}$ (cont.)




Symmetric walk for non-positive sites, non-homogeneous walk with **asymptotically zero drift** $\frac{1}{4x}$ for positive sites.

# Processes with non-integrable jumps

Random walks (adapted processes) with non-itegrable increments

$X_n \sim \left(\mathcal{F}_n\right)_{n\in\mathbb{Z}^+}$ adapted process with $X_0 = 0$;
its increments $\Delta_n = X_{n+1} - X_n = \Delta_n^+ - \Delta_n^-$, where $\Delta_n^\pm \geq 0$.

Key asumptions: fix $\alpha \in (0,1)$ and $\beta > \alpha$. Let, uniformly in $n$, almost surely,

$$\mathsf{E}\big((\Delta_n^-)^\beta \mid \mathcal{F}_n\big) \leq C\,, \tag{L}$$

and, for all $x \geq x_0$, almost surely,

$$\mathsf{E}\big(\Delta_n^+ \mathbb{1}_{\Delta_n^+ \leq x} \mid \mathcal{F}_n\big) \geq c x^{1-\alpha}\,. \tag{R1}$$

$$\mathsf{P}(\Delta_n^+ > x \mid \mathcal{F}_n) \sim x^{-\alpha}\,. \tag{R2}$$

Notice: (R1) implies $\mathsf{E}\big((\Delta_n^+)^\gamma \mid \mathcal{F}_n\big) = \infty$ for every $\gamma > \alpha$.

The regularity condition (R1) cannot be replaced by a moment condition even for random walks, [Chung].

# Transience condition and the rate of escape

**Theorem 1**: *Fix* $\alpha \in (0,1)$ *and* $\beta > \alpha$. *Then (L) & (R1) imply*

$$X_n \to +\infty, \text{ almost surely, as } n \to \infty.$$

**Corollary** : *Fix* $\alpha \in (0,1)$ *and* $\beta > \alpha$. *Then (L) & (R2) imply*

$$\lim_{x \to \infty} \frac{\log X_n}{\log n} = \frac{1}{\alpha}.$$

# First-passage times

For $x \in \mathbb{R}$, define the *first-passage time* for $[x, \infty)$ via

$$\tau_x = \min\{n \in \mathbb{Z}^+ : X_n \geq x\},$$

where $\min \varnothing = \infty$.

**Theorem 2**: *Let $\alpha \in (0,1)$ and $\beta > \alpha$. If (L) and (R1) hold, then for every $x \in \mathbb{R}$ and every $p \in [0, \beta/\alpha)$, we have*

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**Theorem 3**: *Let $\alpha \in (0,1]$ and $\beta > \alpha$. Suppose that for some $C < \infty$, we have, almost surely,*

$$\mathrm{E}\big((\Delta_n^+)^\alpha \mid \mathcal{F}_n\big) \leq C \qquad \text{and} \qquad \mathrm{E}\big((\Delta_n^-)^\beta \mid \mathcal{F}_n\big) = \infty .$$

*Then, for any $x > 0$,*

$$\mathrm{E}\big((\tau_x)^{\beta/\alpha}\big) = \infty .$$

# Last-exit times

For $x \in \mathbb{R}$, define the *last-exit time* from $(-\infty, x]$ via

$$\lambda_x = \max\{n \in \mathbb{Z}^+ : X_n \leq x\}.$$

**Theorem 4**: *Let $\alpha \in (0,1)$ and $\beta > \alpha$. If (L) and (R1) hold, then for every $x \in \mathbb{R}$ and every $p \in [0, (\beta/\alpha) - 1)$, we have*

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$$\mathrm{E}\big((\lambda_x)^p\big) < \infty.$$

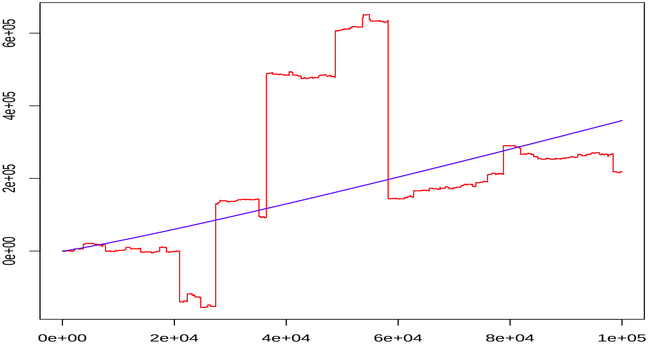
**Theorem 5**: *Let $\alpha \in (0,1]$ and $\beta > \alpha$. Suppose that for some $C < \infty$, $c > 0$, and $x_0 < \infty$, we have, almost surely,*

$$\mathrm{E}\big((\Delta_n^+)^\alpha \mid \mathcal{F}_n\big) \leq C \qquad \text{and} \qquad \mathrm{P}\big(\Delta_n^- > x \mid \mathcal{F}_n\big) \geq c x^{-\beta},$$

*if only $x \geq x_0$. Then, for any $x > 0$ and any $p > (\beta/\alpha) - 1$*

$$\mathrm{E}\big((\lambda_x)^p\big) = \infty.$$

# Random walk with non-integrable increments




Heavy-tailed random walk with $\alpha = 0.9$ and $\beta = 0.94$

# Application: Heavy-tailed walks on strips

Consider a Markov chain $(U_n, V_n)$ on $\mathcal{S}_k = \{0, 1, \ldots, k-1\} \times \mathbb{Z}$ or $\mathcal{S}_\infty = \mathbb{Z}^+ \times \mathbb{Z}$ with jumps

$$\mathrm{P}\big((U_{n+1}, V_{n+1}) = (\ell', x + d) \mid U_n = \ell, V_n = x\big) = \phi(\ell, \ell'; d)\,.$$

[spatial homogeneity in the second coordinate!]

Induced Markov chain $(U_n)_{n\geq 0}$:

$$\mathrm{P}\big(U_{n+1} = \ell' \mid U_n = \ell\big) = \sum_{d\in\mathbb{Z}} \phi(\ell, \ell'; d)\,.$$

**Assumption B:** jumps as above, induced chain $(U_n)_{n\geq 0}$ is *irreducible* and *recurrent*.

# Positive recurrent case

**Theorem 6**: *Assume that* **(B)** *holds and* $U_n$ *is positive-recurrent. Suppose that for some* $\alpha \in (0,1)$, $\beta > \alpha$ *and* $C < \infty$, *a.s.,*

*i)* $\mathrm{E}\big[\big((V_{n+1} - V_n)^-\big)^\beta \mid U_n, V_n\big] \leq C$ ;

*iia) on* $\{U_n = 0\}$,

$$\lim_{x \to \infty} \frac{\log \mathrm{P}\big((V_{n+1} - V_n)^+ > x \mid U_n, V_n\big)}{\log x} = -\alpha \, ;$$

*iib) on* $\{U_n \neq 0\}$, $\mathrm{E}\big[\big((V_{n+1} - V_n)^+\big)^\beta \mid U_n, V_n\big] \leq C$.

*Then, a.s.,* $V_n \to +\infty$ *as* $n \to \infty$*; moreover, a.s.,*

$$\lim_{n \to \infty} \frac{\log V_n}{\log n} = \frac{1}{\alpha} \, .$$

# Null recurrent case

Let $\nu = \min\{n > 0 : U_n = 0\}$ be the first return time to the 0-line.

**Theorem 7**: *Assume that* **(B)** *holds, $U_n$ is null-recurrent such that $\lim_{n\to\infty} \log \mathsf{P}(\nu > n)/\log n = -\gamma$, for some $\gamma \in (0,1]$.*

*Suppose that for some $\alpha \in (0,1)$, $\beta > 0$ and $C < \infty$, a.s.,*

*i)* $\mathsf{E}\big[\big((V_{n+1} - V_n)^-\big)^\beta \mid U_n, V_n\big] \le C$ ;

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*If $\alpha < \gamma(\beta \wedge 1)$, then, a.s., $V_n \to +\infty$ as $n \to \infty$; moreover, a.s.,*

$$\lim_{n\to\infty} \frac{\log V_n}{\log n} = \frac{\gamma}{\alpha} .$$

# Null recurrent case (cont.)

**Theorem 8**: *Assume that* **(B)** *holds,* $U_n$ *is null-recurrent and* $\nu$ *is as in Theorem 7.*

*Suppose that for some* $\alpha, \beta \in (0,1)$, $\delta > 0$ *and* $C < \infty$, *a.s.,*

*i) on* $\{U_n = 0\}$, $\mathsf{E}\big[|V_{n+1} - V_n|^\alpha \mid U_n, V_n\big] \le C$ *;*

*ii) on* $\{U_n \neq 0\}$,

$$\lim_{x\to\infty} \frac{\log \mathsf{P}\big((V_{n+1} - V_n)^- > x \mid U_n, V_n\big)}{\log x} = -\beta\,;$$

*iii) on* $\{U_n \neq 0\}$, $\mathsf{E}\Big[\big((V_{n+1} - V_n)^+\big)^{\beta+\delta} \mid U_n, V_n\Big] \le C$.

*If* $\alpha > \gamma\beta$, *then, a.s.,* $V_n \to -\infty$ *as* $n \to \infty$*; moreover, a.s.,*

$$\lim_{n\to\infty} \frac{\log |V_n|}{\log n} = \frac{1}{\beta}\,.$$

# Heuristics

If $\xi$ has heavy tails, eg., $\mathrm{P}(|\xi| > x) \asymp x^{-\alpha}$, then the sum $S_k = \xi_1 + \cdots + \xi_k$ of $k$ independent copies of $\xi$ is of order $k^{1/\alpha}$. It thus takes about $n^\alpha$ steps to travel distance of order $n$.

[Marcinkiewicz-Zygmund 1937]

In particular, if the return time $\nu = \min\{n > 0 : U_n = 0\}$ satisfies

$$\lim_{n\to\infty} \log \mathrm{P}(\nu > n)/\log n = -\gamma\,,$$

by time $T$ the Markov chain $U_n$ visits the boundary state 0 approximately $T^\gamma$ times.

By time $T$, the total boundary shift is of order $\left(T^\gamma\right)^{1/\alpha} = T^{\gamma/\alpha}$, the bulk shift is of order $T^{1/\beta}$.

# Concluding remarks

- Instead of working with random walks we could work with continuous processes (diffusions) instead.
- Our methods use martingale ideas. An advantage of the martingale approach is that the Markov property is not essential to the proofs. The martingale approach gives an "easy" proof of Pólya's theorem that generalizes broadly.
- Similar methods can also be applied in the **heavy-tailed** setting [HMMW 12].

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# Concluding remarks (cont.)

- Non-homogeneous random walks can be viewed as prototypical **near-critical** stochastic systems, in the sense that small perturbations close to a phase boundary lead to rich variations in behaviour. This study fits within a broad programme of developing methods to study near-critical systems, where classical methods usually fail.
- So the techniques that we developed in this work can be (and have been) applied to other near critical systems with applications in probability and beyond, such as **queueing systems**, **interacting particle systems**, and **processes in random media**.

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# References


- O. Hryniv, I.M. MacPhee, M.V. Menshikov, and A.R. Wade, **Non-homogeneous random walks with non-integrable increments and heavy-tailed random walks on strips, Electron. J. Probab. (2012).**
- O. Hryniv, M.V. Menshikov, and A.R. Wade, **Excursions and path functionals for stochastic processes with asymptotically zero drifts, Stoch. Proc. Appl. (2013).**
- J. Lamperti, **Criteria for the recurrence or transience of stochastic processes I, J. Math. Anal. Appl. (1960).**
- J. Lamperti, **A new class of probability limit theorems, J. Math. Mech. (1962).**
- J. Lamperti, **Criteria for stochastic processes II: passage-time moments, J. Math. Anal. Appl. (1963).**