Periodicity of Markov polling systems in overflow regimes

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Joint work with
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Durham, 3 April 2014

Source paper: **http://pi.vu/B27O**

Iain MacPhee

8 November 1957 - 13 January 2012

K queues **One server**

"**Relative price**" of a customer in queue i vs. j is p_{ij}

Arrival rates: $\lambda_1 \ldots \lambda_K$

Service rates: $\mu_1 \ldots \mu_K$

Load rates: $\rho_i = \lambda_i / \mu_i$

Each $\rho_i < 1$, but $\Sigma\, \rho_i > 1$

Service discipline:

- When the server is at node i its serves the queue $Q_i(t)$ while it is non-empty
- When the current queue (say 1) becomes empty, the server goes to the "most expensive" node, for example to 2 whenever $Q_2(t)/ Q_j(t) > p_{2j}$ $\quad j=3,\ldots,K$

The system will "overflow" but not at *an individual* node!

<u>Our main result</u>*: the service will be <u>*periodic*</u> from some moment of time**

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* K=3

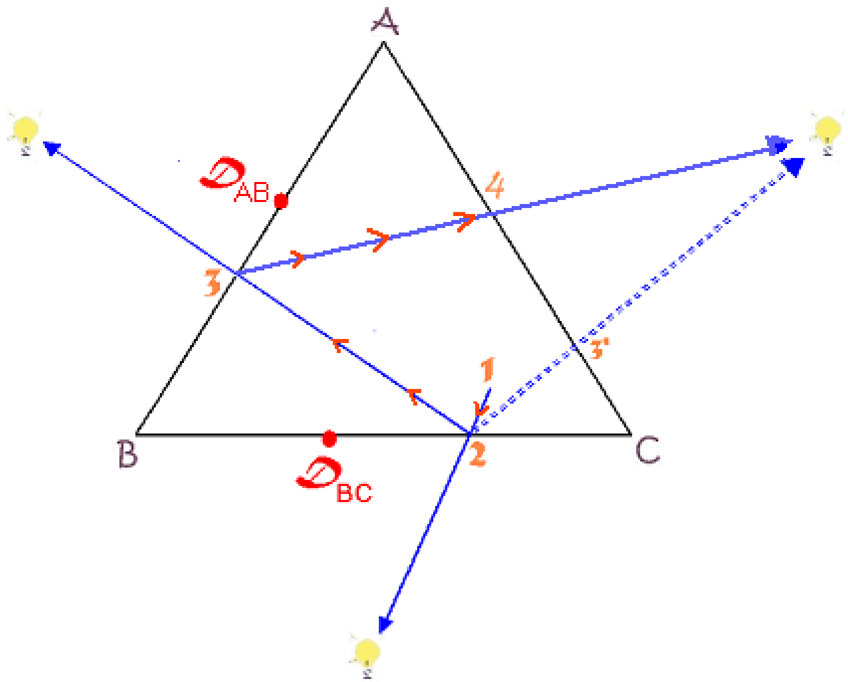
** for almost all configurations of parameters

Approach: to analyze the corresponding dynamical (fluid) system

K=3 from now on

The state of the system can be represented as a point on a 3D simplex, i.e. inside the equilateral triangle *ABC*

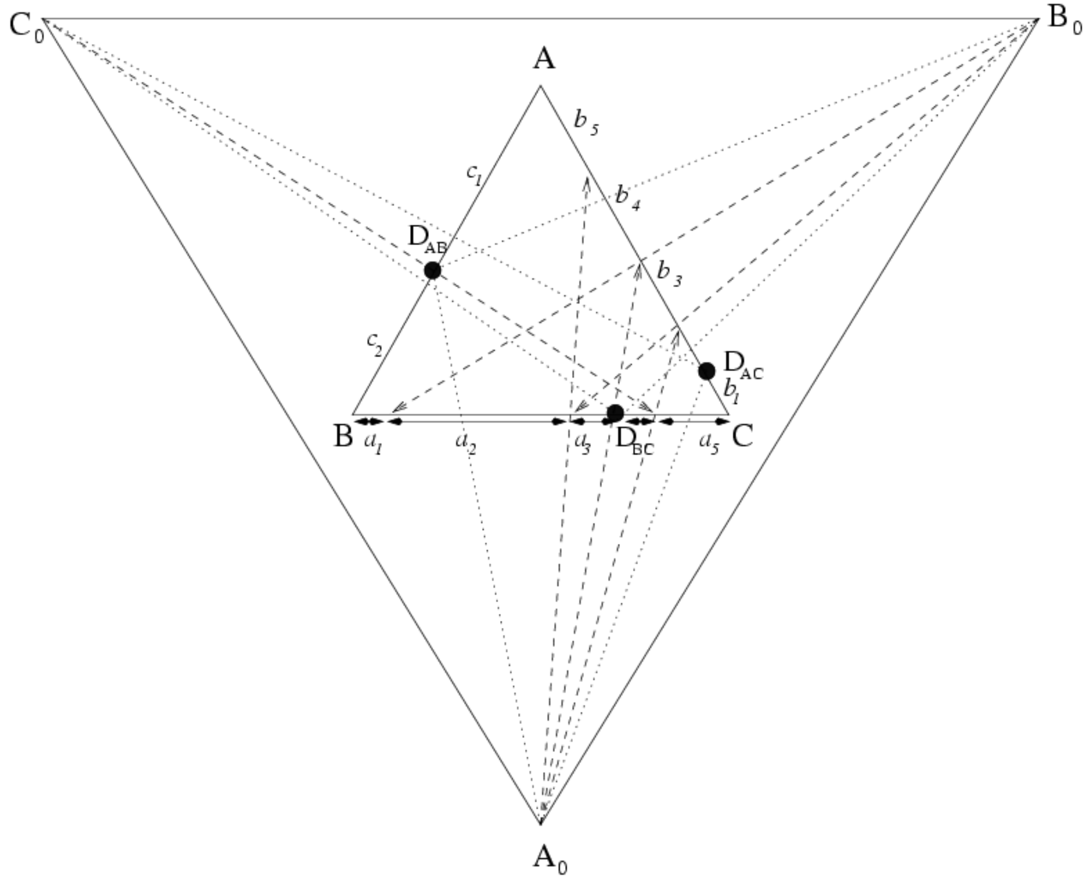
Points on the sides correspond to situations when one of the queues is empty.



There is a *decision point* on each side

Mapping φ: to light sources A_0 B_0 and C_0, depending on the positions of decision points

If each decision point has finitely many pre-images under φ, then the corresponding dynamical system will be periodic (follows from *pigeonhole principle*)



For this configuration, the only period will be [*cbabacaba*] – with length *9*.

Theorem 1

Assume each of the decision points D_{AB}, D_{BC}, D_{CA} has finitely many pre-images under □.

- the dynamical system is periodic. At most **4** distinct periods (*up to rotations*);
- the stochastic polling system is also periodic and has the same periods as the dynamical one.

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Theorem 2

… for almost all configurations of the parameters (e.g. p_{12} has a continuous conditional distribution on some domain when the other parameters are fixed) each of the decision points D_{AB}, D_{BC}, D_{CA} has finitely many pre-images under φ.

Theorem 3

There are **uncountably many** these "bad" configurations of decision points. For them:

- some trajectories of the dynamical system are aperiodic.
- $0 < P$ (the polling system is aperiodic) < 1.

Key properties of the dynamical system:

LINEARITY (projection) PRINCIPLE

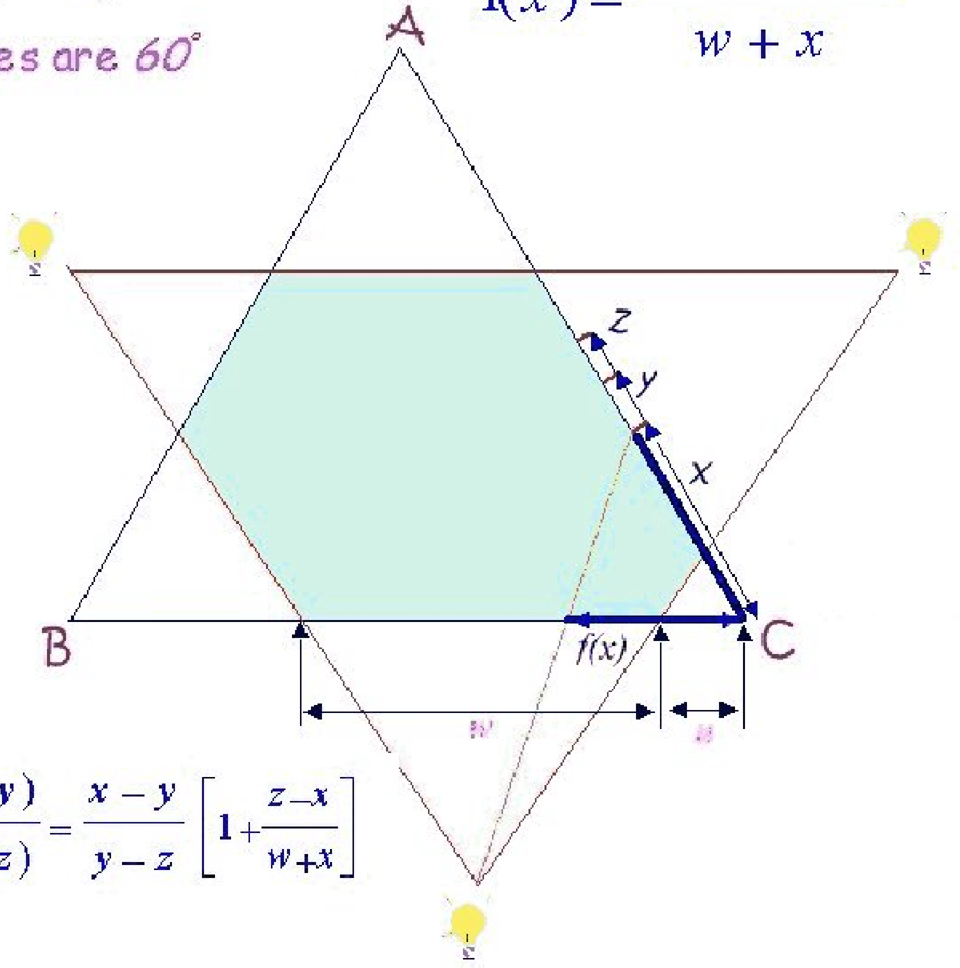
Second equilateral triangle PRINCIPLE

Uniform CONTRACTION PRINCIPLE

$$\mu_1 = \mu_2 = \mu_3 = 1$$

All angles are 60°

$$f(x) = \frac{x\,(w+u)}{w+x}$$



$$\frac{f(x) - f(y)}{f(y) - f(z)} = \frac{x-y}{y-z}\left[1 + \frac{z-x}{w+x}\right]$$

How to justify the approximation?

- The lengths of the queues increase exponentially, at least after some random time
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Let

$$f(t)=\sum_{i=1}^{K}\frac{Q_i(t)}{\mu_i}$$

Observe that when we serve node *j*

$$E\left(f(t+dt)-f(t)\,|\,\Im(t)\right)=\frac{\sum_{i=1}^{K}\lambda_i dt}{\mu_i}-\frac{\mu_j dt}{\mu_j}=[\sum_{i=1}^{K}\rho_i-1]dt=\eta dt>0$$

Hence *f* is a sub-martingale

Suppose the server at time τ_j has just cleared out node 3. Set $f_j = f(\tau_j)$

Let $X = Q_1(\tau_j)$, $Y = Q_2(\tau_j)$, and $Z = Q_3(\tau_j) = 0$ be the queue sizes at 1, 2, and 3.
Suppose w.l.o.g. $X / Y > p_{12}$ so the server ought to move to node 1.

Let τ_{j+1} be the time when the queue at 1 is emptied. Then, if the system did not have any randomness in it,

$$X \mapsto 0, \quad Y \mapsto Y + \lambda_2 \frac{X}{\mu_1 - \lambda_1}, \quad Z \mapsto \lambda_3 \frac{X}{\mu_1 - \lambda_1}$$

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yielding

$$\frac{f_{j+1}}{f_j} = \frac{\frac{1}{\mu_2}\left(Y + \lambda_2 \frac{X}{\mu_1 - \lambda_1}\right) + \frac{1}{\mu_3}\frac{\lambda_3 X}{\mu_1 - \lambda_1}}{\frac{X}{\mu_1} + \frac{Y}{\mu_2}} = \frac{\frac{X}{\mu_1}\left(\frac{\rho_2 + \rho_3}{1 - \rho_1}\right) + \frac{Y}{\mu_2}}{\frac{X}{\mu_1} + \frac{Y}{\mu_2}}$$

$$= 1 + \frac{\rho_1 + \rho_2 + \rho_3 - 1}{1 - \rho_1}\left(1 + \frac{\mu_1}{\mu_2}\frac{Y}{X}\right)^{-1} > 1 + \frac{\eta}{1 - \rho_1}\left(1 + \frac{\mu_1}{\mu_2}\frac{1}{p_{12}}\right)^{-1} \geq 1 + \nu$$

Thus, for the dynamical system we would have $f_j \propto (1+\nu)^j$

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(Recall: $f(t) = \sum_{i=1}^{K} \frac{Q_i(t)}{\mu_i}$)

Now since

$$f_j = f(\tau_j) = \frac{X}{\mu_1} + \frac{Y}{\mu_2} \leq \max\left(1, \frac{1}{p_{12}}\right) \times \left(\frac{X}{\mu_1} + \frac{X}{\mu_2}\right) \leq CX$$

where

$$C = \max\left(1, p_{12}, p_{21}, p_{13}, p_{31}, p_{23}, p_{32}\right) \times \max\left(\mu_1^{(-1)} + \mu_2^{(-1)}, \mu_1^{(-1)} + \mu_3^{(-1)}, \mu_3^{(-1)} + \mu_2^{(-1)}\right)$$

the length of the most expensive queue goes to infinity, as long as $f_j \rightarrow \infty$.

Deviations of the stochastic system

Obtain exponential bounds on the probability that the j-th service time $\tau_{j+1}-\tau_j$ deviates by more $\left(\frac{X}{\mu_1-\lambda_1}\right)^{2/3}$ from its expected value of $\frac{X}{\mu_1-\lambda_1}$.

We obtain similar bounds for the increments of the other two queues.

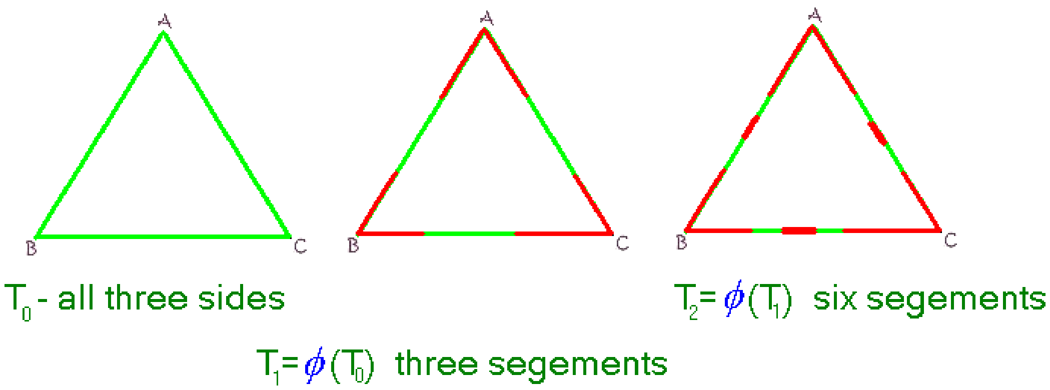
There is j_0 such that for all $j>j_0$ in fact $f_{j+1}>(1+\nu/2)f_j$ with probability exponentially($-j$) close to 1.

Let $\delta>0$ be smaller than the length of the smallest interval created by the set

P={**all pre-images of decision points**}

After j_0, which we might choose large enough, the "**lifetime deviation**" of the stochastic system from the fluid one is smaller than $\delta/2$ with probability also close to 1.

Let T_0=all the sides of the triangle ABC; and $T_n=\varphi(T_{n-1})$.



- $T_n \subseteq T_{n-1}$
- for $n\geq 1$ every T_n consists of at most 3×2^n segments, 2^n on each side of the triangle.

total Lebesgue measure of segments in $T_n \to 0$ as $n \to \infty$.

We can choose n_0 so large that

for all $n \geq n_0$ distance(T_n,P) > $\delta/2$

Let x_j be the state of the stochastic system at time τ_j, and
let y_j be the closest to x_j point of T_{j-j_0}, possibly x_j itself.

Let x_j be the state of the stochastic system at time τ_j, while $y_j = \varphi(y_{j-1})$

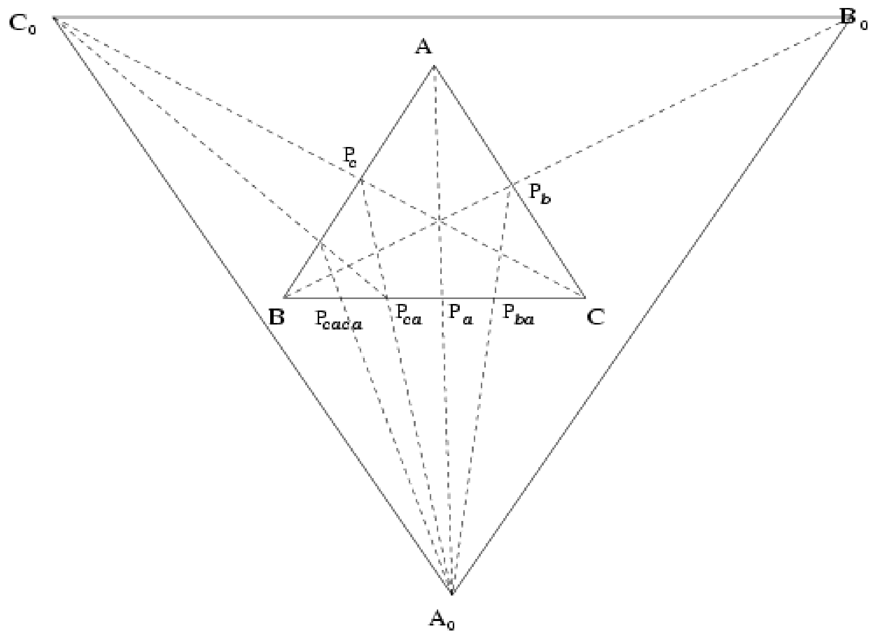
Then as j grows, the distances between x_j and y_j decay exponentially (contraction principle), unless there's a decision point between them at some time j'.

However then latter is impossible
(conditioned on not deviating by more than δ).

As a result, y_j "drags" x_j from the same to the same side of the triangle ABC.

And the deterministic dynamical system is periodic!

Construction of "bad" decision points TRIPLES



BP_{caca} = "a : 000", $P_{caca}P_{ca}$ ="a: 001", $P_{ca}P_a$ ="a :01",

P_aP_{ba} ="a : 10", $P_{ba}C$ ="a : 11"; CP_b ="b : 0", P_bA ="b : 1", etc.

Algebraic representation of mapping φ.

Each point x on side $a \equiv BC$ can be written as an infinite sequence of 0's and 1's.

e.g. $x = a$: 0 1 0 1 1 1 0…

then $\varphi(x) =$ ***b*: 0** 1 0 1 0 0 0 1…

or $\varphi(x) =$ ***c*: 1** 1 0 1 0 0 0 1…

Set decision points to be y

$D_{BC} = a$: $qrq\ldots$ ("…" - variable pattern)

$D_{CA} = b$: $0100000\ldots$ ("…" - all zeros)

$D_{AB} = c$: $1010100000\ldots$ ("…" - all zeros)

where $q = \underline{1001}$
and $r = \underline{0110}$.

The sequence for D_{BC} can be written as $y = y_1 y_2 y_3 \ldots$ where $y_i =\in \{q, r\}$.

$r < q$

Lemma:

If y satisfies the following properties

(a) if $y_k = r$ then $y_{k+1} y_{k+2} y_{k+3} \ldots > y$

(b) if $y_k = q$ then $y_{k+1} y_{k+2} y_{k+3} \ldots < y$

then D_{BC} has infinitely many pre-images under φ

Such sequences may be "*easily*" constructed using rational approximations of irrational numbers