Predator-prey dynamics on infinite trees

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Partially based on joint work with Ghurumuruhan Ganesan.

The SIR spreading model

A disease is propagating along the edges of a graph G. A vertex may either be

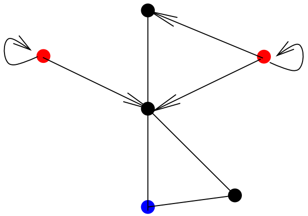
- (S)usceptible
- (I)nfected and infectious
- (R)ecovered.

$\Rightarrow$ *rumor spreading, epidemic, prey and predator, information dissemination ...*

Standard SIR dynamics

Markov process on $\mathcal{X} = \{S, I, R\}^V$:

- *a* (S)*-vertex becomes* (I) *at rate* λ *times the number of* (I)*-neighbors,*
- *a* (I)*-vertex becomes* (R) *at rate* 1*.*



Chase-escape model

Kordzakhia (2005)

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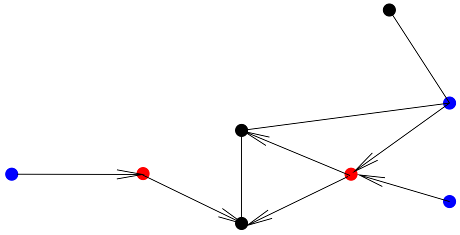

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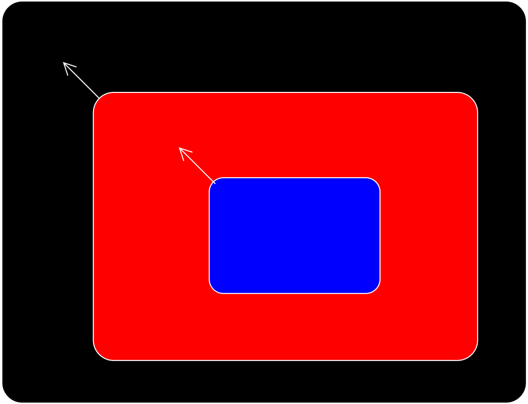


⇒ *Nested propagation ...*

Chase-Escape Process on a Tree

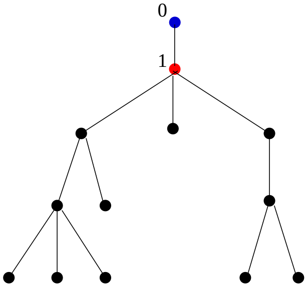
Propagation on trees

In this talk : SIR model on short time scale.

A simple model : the underlying graph T is an infinite tree.

$\Rightarrow$ *Reasonable approximation for locally tree-like graphs.*

Our initial condition :



Absorbing States

$\Rightarrow$ The states without (I)-vertices are absorbing.

Either :

(i) at some finite time, the process reaches an absorbing state.
(I)-vertices die out.

(ii) (I)-vertices survive indefinitely.

(R)-vertices = vertices that have been infected.

Assumption on the tree

Upper growth rate :

$$d = \limsup_{k\to\infty} |V_k|^{1/k} \in (1, \infty).$$

where V_k = set of vertices at distance k from the root.

Lower d-ary $\simeq$ for large k, the distance-k tree contains a $(d - o(1))^k$-ary tree.

Satisfied a.s. for Galton-Watson tree with mean number of offspring $d > 1$ *conditioned on non-extinction.*

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For which value

$$q_T(\lambda) = \mathbb{P}_\lambda(\text{exctinction}) < 1 \text{ ?}$$

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Theorem (Kordzakhia (2005) - d-ary tree)
Let

$$\lambda_1 = 2d - 1 - 2\sqrt{d(d-1)} \qquad (\sim \frac{1}{4d}).$$

(i) *If $\lambda < \lambda_1$ and T has upper growth rate d then $q_T(\lambda) = 1$.*

(ii) *If $\lambda > \lambda_1$ and T is lower d-ary then $q_T(\lambda) < 1$.*

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$\Rightarrow$ For the standard SIR : $\lambda_1 = 1/(d-1)$.

Annealed survival probability

Assume T is a Galton-Watson tree with offspring distribution P of mean $d > 1$.

$$q(\lambda) = \mathbb{E}' q_T(\lambda) = \mathbb{P}'_\lambda(\text{exctinction}).$$

Theorem

If the offspring distribution has finite second moment then for all $\lambda_1 < \lambda < 1$,

$$c_0 \omega^3 e^{-\frac{(1-\lambda_1)\pi}{2(d(d-1))^{1/4}}\omega^{-1}} \leqslant 1 - q(\lambda) \leqslant c_1 e^{-\frac{(1-\lambda_1)\pi}{2(d(d-1))^{1/4}}\omega^{-1}},$$

with

$$\omega = \sqrt{\lambda - \lambda_1}.$$

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⇒ For the standard SIR : $1 - q(\lambda) = \Theta(\lambda - 1/(d-1))_+$.

⇒ Similar result for Brunet-Derrida's model of branching random walk killed below a linear barrier.

Subcritical Regime

If $0 < \lambda \leqslant \lambda_1$, let Z be the total infected population on the GWT (number of (R)-vertices in absorbing state).

Tail exponent

$$\gamma(\lambda) = \sup\{u \geqslant 0 : \mathbb{E}'_\lambda[Z^u] < \infty\},$$

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$\Rightarrow$ For the standard subcritical SIR : $\gamma(\lambda) = \gamma_P$.

Computation of moments

It is even possible to compute by recursion the moments of Z on the GWT. The first moment is

Theorem
If $0 < \lambda \leqslant \lambda_1$ *and* $\Delta = \lambda^2 - 2\lambda(2d-1) + 1$, *then*

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Corollary
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$\Rightarrow$ For the standard subcritical SIR : $\mathbb{E}'_{1/(d-1)}[Z] = \infty$.

Birth-and-Assassination Process

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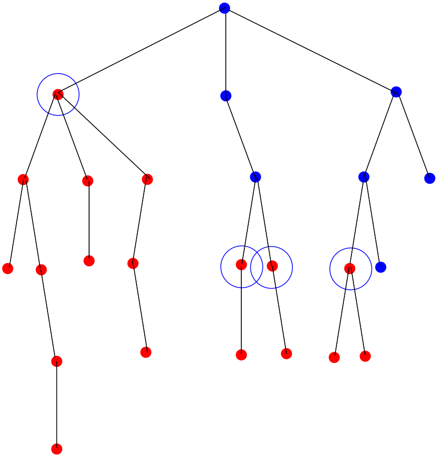
The root produces children at rate λ.

Each new vertex produces children at rate λ.

The root is at risk at time 0 and dies at time D, an exponential variable with parameter 1.

Other vertices are at risk when its ancestor dies, and dies after an independent copy of D.

Birth-and-Assassination Process



$\Rightarrow$ Scaling limit as $d \to \infty$ of the chase-escape model with intensity λ/d. Easier formulas.

Phase transition

$$q(\lambda) = \mathbb{P}_\lambda(\text{exctinction}).$$

Theorem (Aldous & Krebs (1990))

(i) *If* $0 < \lambda < 1/4$,

$$q(\lambda) = 1.$$

(ii) *If* $\lambda > 1/4$,

$$q(\lambda) < 1.$$

Subcritical phase

For $0 < \lambda < 1/4$, Z = total population in the BA process.

Tail exponent

$$\gamma(\lambda) = \sup\{u \geqslant 0 : \mathbb{E}_\lambda[Z^u] < \infty\}.$$

Theorem

For all $0 < \lambda \leqslant 1/4$,

$$\gamma(\lambda) = \frac{1+\sqrt{1-4\lambda}}{1-\sqrt{1-4\lambda}}.$$

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(ii) *If* $\lambda \in (0, 2/9)$,

$$\mathbb{E}_\lambda[Z^2] = \frac{2}{3\sqrt{1-4\lambda}-1}.$$

(iii) *If* $\lambda \in (0, 3/16)$,

$$\mathbb{E}_\lambda[Z^3] = \cdots$$

Corollary

For $\lambda = 1/4$, $\mathbb{E}_{1/4}[Z] = 2$ *and* $q(1/4) = 1$.

Recursive Distributional Equation

$Y(t)$ = the total population given that the root dies at time t.

If D is an exponential variable with parameter 1, independent of Y

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First Moment

Assume that $\mathbb{E}Y(t) < \infty$ for all $t \geqslant 0$. Taking expectation, we get

$$\begin{aligned}\mathbb{E}Y(t) &= 1 + \lambda \int_0^t \int_0^\infty \mathbb{E}Y(x+s)e^{-s} ds dx \\ &= 1 + \lambda \int_0^t e^x \int_x^\infty \mathbb{E}Y(s)e^{-s} ds dx\end{aligned}$$

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If $0 < \lambda \leqslant 1/4$, the roots of $X^2 - X + \lambda = 0$ are real $0 < \alpha \leqslant \beta \cdots$

$$\mathbb{E}Y(t) = e^{\alpha t}.$$

If $\lambda > 1/4$ no admissible solution of the integral equation.

Other moments

It is possible to generalize this argument to compute all moments of Z.

Everything boils down to linear second order differential equations.

$+$ extend to the case GWT.

Probability of Extinction

$$q(\lambda) = \mathbb{P}_\lambda(\text{exctinction}).$$

Theorem
For all $1/4 < \lambda < 1$,

$$c_0 \omega^3 e^{-\frac{\pi}{2}\omega^{-1}} \leqslant 1 - q(\lambda) \leqslant c_1 \omega^{-1} e^{-\frac{\pi}{2}\omega^{-1}},$$

with

$$\omega = \sqrt{\lambda - 1/4}.$$

Recursive Distributional Equation

$$Q_\lambda(t) = \mathbb{P}_\lambda\left(\text{exctinction}|\text{root dies at time } t\right).$$

$$q(\lambda) = \int_0^\infty Q_\lambda(t)e^{-t}dt.$$

If $(D_i)_{i\geqslant 1}$ iid exponential variables with parameter 1, $\{\xi_i\}_{i\geqslant 1}$ independent a Poisson point process of intensity λ :

$$Q_\lambda(t) = \mathbb{E}\prod_{\xi_i\leqslant t} Q_\lambda(t-\xi_i+D_i).$$

$\Rightarrow$ To get extinct, the subtrees of all children of the root must get extinct.

A Non-Linear ODE

Through

$$x(t) = -\ln Q_\lambda(t),$$

using Lévy-Khinchin formula, we get

$$x'' - x' + \lambda - \lambda e^{-x} = 0,$$

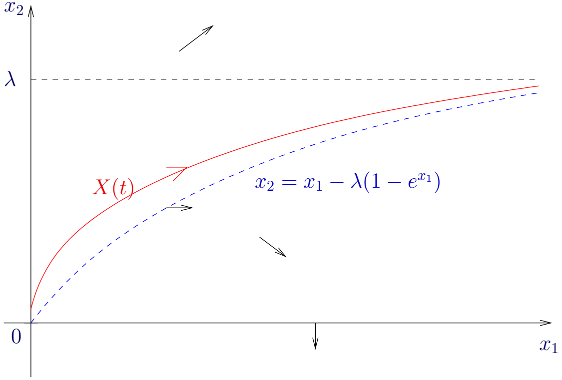
with $x(0) = 0$.

$\Rightarrow$ Near criticality, $\lambda = 1/4$, it is possible to study this type of ODE, (*Brunet-Derrida (1997), Mueller-Mytnik-Quastel (2011)*).

Phase Diagram

We have $X' = F(X)$ with $X = \begin{pmatrix} x \\ x' \end{pmatrix}$ and

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 - \lambda(1 - e^{x_1}) \end{pmatrix}$$



$\Rightarrow$ Near criticality, $\lambda = 1/4$, we linearize the ODE at the origin.

On Finite Graphs

Rumor Scotching Process

A variant of the chase-escape process (*rumor scotching process*) :

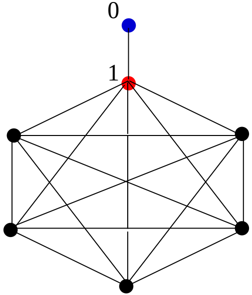
- *a* (I)*-vertex becomes* (R) *at rate* 1 *times the number of neighboring* (R)*-vertices that have infected the vertex.*

$\Rightarrow$ The rumor is confidential.

$\Rightarrow$ On trees, with our initial condition, the CE and RS processes are equal.

On the Complete Graph

Infection rate is λ/n.



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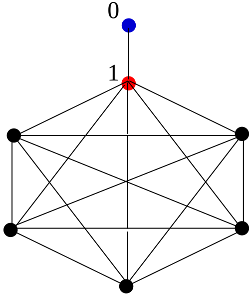

Absorbing states = no (I)-vertex.

Z_n = total population of infected vertices.

The scaling limit of the process as $n \to \infty$ is the BA process.

Heuristic for the Phase Transition

One can guess that Z_n/n converges weakly to W with

$$W \overset{d}{=} q\delta_0 + (1-q)\delta_1,$$

with

$$q(\lambda) = \mathbb{P}_\lambda\left(\text{extinction in the BA process}\right).$$

$\Rightarrow$ Either quick extinction or total invasion.

$\Rightarrow$ For the standard SIR : $W \overset{d}{=} q\delta_0 + (1-q)\delta_{1-q}$.

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- Similarly, uniform random graphs with given degree sequence have Galton-Watson trees has local limit.

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- Bring back the particles !

- Chase-escape process on a lattice ?

- Long time scale on the complete graph : analog of Kermack-McKendrick ODE system ?

Thank you for your attention!

Extinction probability and total progeny of predator-prey dynamics on infinite trees, with Ghurumuruhan Ganesan. Preprint, arXiv :1210.2883.

On the birth-and-assassination process, with an application to scotching a rumor in a network. Electronic Journal of Probability, 2014-2030, 2008.