Component structure of the vacant set induced by a random walk on a random graph

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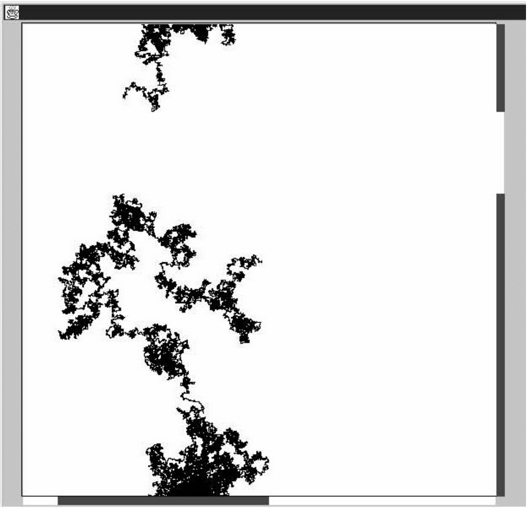
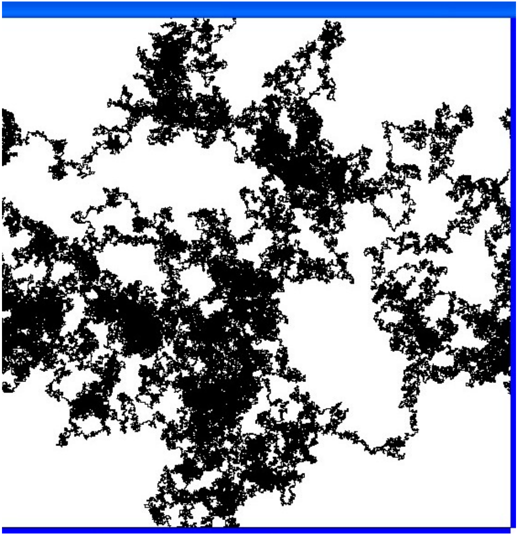
- Vacant set definition and results
- Introduction: random walks and cover time
- $G_{n,p}$ vacant set
- Random r-regular graphs, vacant set

What the a vacant set of a random walk?



Random walk on 600 $\times$ 600 toriodal grid. Black visited, white unvisited.

What is the component structure of vacant set?



Notation

Finite graph $G = (V, E)$.

$\mathcal{W}_u$ Simple random walk on G, starting at $u \in V$

The vacant set
$\mathcal{R}(t)$ Set of vertices unvisited by $\mathcal{W}_u$ up to time t
$\Gamma(t)$ Sub-graph of G induced by vacant set $\mathcal{R}(t)$

Can think of vacant set $\mathcal{R}(t)$ as coloured red, and visited vertices $\mathcal{B}(t)$ as colored blue

How large is $\mathcal{R}(t)$?
What is the likely component structure of $\Gamma(t)$?

Evolution of vacant set

As the walk progresses $\Gamma(t)$ is reduced from the whole graph G to a graph with no vertices

In the context of sparse random graphs, as $\mathcal{R}(t)$ gets smaller, $\Gamma(t)$ will get sparser and sparser. (Small sets don't induce many edges)

One might expect that at some time $\Gamma(t)$ will break up into small components
This is basically what we prove. It is a sort of random graph process in reverse

We say that $\Gamma(t)$ is sub-critical at step t, if all of its components are of size $O(\log n)$

We say that $\Gamma(t)$ is super-critical at step t, if it has a unique giant component, (of size $\Theta(\mathcal{R}(t))$) and all other components are of size $O(\log n)$

In the cases we consider there is a t^*, which is a (**whp**) threshold for transition from super-criticality to sub-criticality

Vacant set of $G_{n,p}$

We assume that

$$p = \frac{c \log n}{n}$$

where $(c-1)\log n \to \infty$ with n, and $c = n^{o(1)}$. Let

$$t(\epsilon) = n\left(\log\log n + (1+\epsilon)\log c\right)$$

Theorem

Let $\epsilon > 0$ be a small constant
Then **whp** *we have*
(i) $\Gamma(t)$ is super-critical for $t \leq t(-\epsilon)$
(ii) $\Gamma(t)$ is sub-critical for $t \geq t(\epsilon)$

Giant component of $\mathcal{R}(t)$ until $t > n \log\log n$
Cover time $T_{cov} \sim n \log n$ when $c > 1$ constant

Random graphs $G_{n,r}$

For $r \geq 3$, constant, let

$$t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2}\, n$$

Theorem

Let $\epsilon > 0$ be a small constant. Then **whp** *we have*

(i) $\Gamma(t)$ *is super-critical for* $t \leq (1-\epsilon)t^*$

(ii) *For* $t \leq (1-\epsilon)t^*$*, size of giant component is* $\Omega(n)$

(iii) $\Gamma(t)$ *is sub-critical for* $t \geq (1+\epsilon)t^*$

e.g. for 3-regular random graphs $r = 3$, and $t^* = (6\log 2)\, n$
Giant component until $t^*(6\log 2)n$
Cover time $T_{cov} \sim 2n\log n$

Previous Work

Benjamini and Sznitman; Windisch:
Considered the infinite d-dimensional torus $d \geq 3$, and discrete torus for large d

Černy, Teixeira and Windisch:
Considered random r-regular graphs $G_{n,r}$
They show sub-criticality for $t \geq (1+\epsilon)t^*$
and existence of a unique giant component for $t \leq (1-\epsilon)t^*$
These proofs use the concept of random interlacements of continuous time random walks

Our proof: Discrete time

- Simple. Based on established random graph results
- Gives results for $G_{n,p}$
- Completely characterizes the component structure
- Proves that in the super-critical phase $t \leq t^*$, the second largest component of $G_{n,r}$ has size $O(\log n)$ **whp**
  Gives the small tree structure of $\Gamma(t)$

Subsequent Work: Černy, Teixeira and Windisch:
Consider random r-regular graphs $G_{n,r}$
Investigate scaling window around t^* using annealed model

Proof technique: r-regular r.g's

- Use walk to reveal the graph: Annealed model
- Estimate un-visit probability of vertices by walk and hence size and degree sequence $\boldsymbol{d}$ of vacant set $\mathcal{R}(t)$
- Graph $\Gamma(t)$ induced by vacant set $\mathcal{R}(t)$ is random
- Given degree sequence $\boldsymbol{d}$ of $\Gamma(t)$, use Molloy-Reed condition for existence and size of giant component
- Count small tree components

Cover time T_{cov} of random walk on graph G

T_{cov} is the maximum expected time, over all start vertices u, for a random walk $\mathcal{W}_u$ to visit all vertices of G.

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3. Web-graphs $G(m,t)$ where $m \geq 2$
   $$T_{cov} \sim \frac{2m}{m-1} t \log t$$

Directed graphs: random digraphs $D_{n,p}$

The main challenge for $D_{n,p}$, was to obtain the stationary distribution

Theorem

Let $np = d \log n$ *where* $d = d(n)$, *and let* $m = n(n-1)p$

Let $\gamma = np - \log n$, *and assume* $\gamma = \omega(\log \log n)$

Then **whp**, *for all* $v \in V$,

$$\pi_v \sim \frac{\deg^-(v)}{m},$$

and

$$T_{cov} \sim d \log\left(\frac{d}{d-1}\right) n \log n$$

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  where R_v is expected number of returns to v during a suitable mixing time
- Waiting time of first visit to v tends to geometric distn, success probability $p_v \sim \pi_v/R_v$

Summary: Unvisit Probability

Let T_{mix} be a suitable mixing time of the walk.
Let π_v denote the stationary distribution of v.
Let R_v denote the expect number of returns to v by the walk $\mathcal{W}_v$ in the time T_{mix}.

Then **Unvisit Probability**

$$\mathbf{Pr}(\mathcal{W}_u(\tau) \neq v : \tau = T_{mix}, \ldots, t) \sim e^{-t\pi_v/R_v}$$

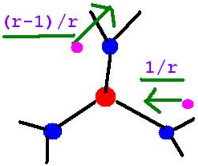
True under assumptions that hold for many random graph models

For random graphs we can estimate R_v accurately from the graph structure for most vertices, and bound it suitably for all vertices

Example: r-regular random graphs

How to calculate R_v for random r-regular graphs ?

If v is tree-like (not near any short cycles) then $R_v \sim \frac{r-1}{r-2}$



Same as: biassed random walk on the half line $(0, 1, 2, ...)$

$\mathbf{Pr}(\text{ go left }) = \frac{1}{r}$, $\mathbf{Pr}(\text{ go right }) = \frac{r-1}{r}$

Example: r-regular random graphs

- $\pi_v = 1/n$
- T_{mix} the mixing time $O(\log n)$
- Most vertices are locally tree-like
  For such vertices $R_v \sim (r-1)/(r-2)$, expected number of returns to start in infinite r-regular tree

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- $\mathcal{R}(t)$ set of vertices not visited by walk at step t
- Size of set of unvisited vertices $\mathcal{R}(t) \sim ne^{-t(r-2)/(r-1)n}$

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- $\mathcal{R}(t)$ set of vertices not visited by walk at step t
- Size of set of unvisited vertices $\mathcal{R}(t) \sim ne^{-t(r-2)/(r-1)n}$
- We know the size of $\mathcal{R}(t)$, the **vacant set**
- Now we need to find the structure of $\mathcal{R}(t)$

Component structure of vacant set of $G_{n,p}$

Distribution of edges in $\Gamma(t)$

Lemma

Consider a random walk on $G_{n,p}$
Conditional on $N = |\mathcal{R}(t)|$, $\Gamma(t)$ is distributed as $G_{N,p}$.

Proof This follows easily from the principle of deferred decisions. We do not have to expose the existence or absence of edges between the unvisited vertices of $\mathcal{R}(t)$ □

Thus to find the super-critical/ sub-critical phases, we only need high probability estimates of $|\mathcal{R}(t)|$ as t varies

This, we know how to do, from our work on cover time of random graphs

Size of vacant set $\mathcal{R}(t)$ in $G_{n,p}$

Analysis of $G_{n,p}$ is for $np = c \log n$

whp

1. $\mathbf{E}(|\mathcal{R}(t)|) \sim \sum_v e^{-t\pi_v/R_v}$

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1. $\mathbf{E}(|\mathcal{R}(t)|) \sim \sum_v e^{-t\pi_v/R_v}$
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   Thus $\pi_v \sim 1/n$
3. $Rv = 1 + o(1)$ for all $v \in V$

Size of vacant set

$$\mathbf{E}(|\mathcal{R}(t)|) \sim ne^{-(1+o(1))t/n}.$$

We use Chebyshev to show that $|\mathcal{R}(t)|$ is concentrated.

Size of 'giant' component

- Threshold criteria for random graph $G_{N,p}$ is $Np \sim 1$
- Recall that $t_\theta = n(\log\log n + (1+\theta)\log c)$ So, at t_θ,

$$\mathbf{E}(|\mathcal{R}(t_\theta)|p) \sim \frac{1}{c^\theta}$$

- When $\theta = 0$, then $\mathbf{E}(|\mathcal{R}(t_\theta)|p) \sim 1$
- The threshold t^* occurs at around

$$t^* \sim n(\log\log n + \log c)$$

- Size of giant is order $|\mathcal{R}(t_\theta)|$. As $t \to t^*$ from below, Size of 'giant' is order $1/p = n/(c\log n)$. i.e. $|\mathcal{R}(t^*)| \sim 1/p$
- Above t^* max component size collapses to $O(\log n)$

Component structure of vacant set of random graphs $G_{n,r}$ for $r \geq 3$, constant.

Reminder: Vacant set of r-regular random graphs

- Most vertices are locally tree-like
  For such vertices $R_v \sim (r-1)/(r-2)$, expected number of returns to start in infinite r-regular tree

$$\mathbf{Pr}(v \text{ unvisited in } T_{mix}, \ldots, t) \sim e^{-t(r-2)/(r-1)n}$$

- A similar upper bound can be obtained for the non-tree-like vertices
- Size of vacant set $\mathcal{R}(t) \sim ne^{-t(r-2)/(r-1)n}$

Let

$$t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2}\, n.$$

Theorem

Let $\epsilon > 0$ be a small constant. Then **whp** *we have*

(i) $\Gamma(t)$ *is super-critical for* $t \leq (1-\epsilon)t^*$,

(ii) *For* $t \leq (1-\epsilon)t^*$, *size of giant component is* $\Omega(n)$

(iii) $\Gamma(t)$ *is sub-critical for* $t \geq (1+\epsilon)t^*$ *and*

Proof outline for r-regular random graph

- Generate the graph in the configuration model using the random walk
- Graph $\Gamma(t)$ induced by vacant set $\mathcal{R}(t)$ is random
- Estimate un-visit probability of vertices to find size of $\mathcal{R}(t)$
- Estimate degree sequence $\boldsymbol{d}$ of $\Gamma(t)$in the configuration model, using size of vacant set $\mathcal{R}(t)$, and number of unvisited edges $\mathcal{U}(t)$
- Given the degree sequence $\boldsymbol{d}$ of $\Gamma(t)$, we can use Molloy-Reed condition for existence of giant component in a random graph with fixed degree sequence
- Estimate number of small trees in configuration model

Degree sequence of $\Gamma(t)$

Vacant set size. $|\mathcal{R}(t)| = (1+o(1))N_t$ where $N_t = ne^{-\frac{(r-2)t}{(r-1)n}}$

Vertex degree. Let $D_s(t)$ the number of unvisited vertices of $\Gamma(t)$ of degree s in $\Gamma(t)$ (ie with $\boldsymbol{r-s}$ visited neighbours)
For $0 \le s \le r$, and for ranges of t given below, **whp**

$$D_s(t) \sim N_t \binom{r}{s} p_t^s \, (1-p_t)^{r-s}$$

where $p_t = e^{-\frac{(r-2)^2}{(r-1)r}\frac{t}{n}}$

Range of validity. $\tau_{r-s} \ll t \le (1-\epsilon)t_s$ where $\tau_0 = 0$,

$$\tau_{r-s} = n^{1-1/(r-s)}, \qquad t_s = \frac{(r-1)r}{(r-2)(s(r-2)+r)} \cdot n \log n.$$

Uniformity

Lemma

Consider a random walk on G_r. Conditional on $N = |\mathcal{R}(t)|$ and degree sequence $\mathbf{d} = d_{\Gamma(t)}(v), v \in \mathcal{R}(t)$, then $\Gamma(t)$ is distributed as $G_{N,\mathbf{d}}$, the random graph with vertex set $[N]$ and degree sequence $\mathbf{d}$.

Proof Basic idea: Reveal G_r using the random walk.
Suppose that we condition on $\mathcal{R}(t)$ and the *history of the walk*, $\mathcal{H} = (W_u(0), W_u(1), \ldots, W_u(t))$. If G_1, G_2 are graphs with vertex set $\mathcal{R}(t)$ and if they have the same degree sequence then substituting G_2 for G_1 will not conflict with $\mathcal{H}$.
Every extension of G_1 is an extension of G_2 and vice-versa. □

Thus we only need:
Good model of component structure of $G_{N,\mathbf{d}}$
High probability estimates of the degree sequence $D_s(t)$ of $\Gamma(t)$.

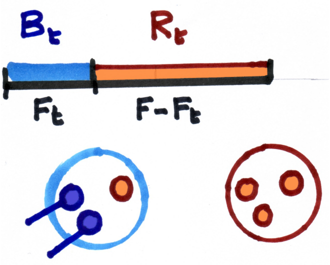
Main variables

By calculating un-visit probabilities in various ways, we can estimate the size at step t of

- $\mathcal{R}(t)$ the set of unvisited vertices
- $\mathcal{U}(t)$ the set of unvisited edges
- $D_s(t)$ the number of unvisited vertices of degree s in $\Gamma(t)$ ie number of unvisited vertices with $r - s$ edges incident with visited vertices $\mathcal{B}(t)$

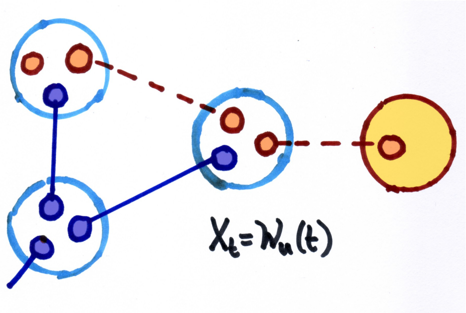
Annealed process

We use the random walk to generate the graph in the configuration model as a random pairing F



- B_t blue conifg. points at step t
  which form discovered pairing F_t
- R_t red conifg. points at step t
  This will form un-generated pairing $F - F_t$
- Visited vertices may have config. points in R_t, corresponding to unexplored edges

Next configuration pairing

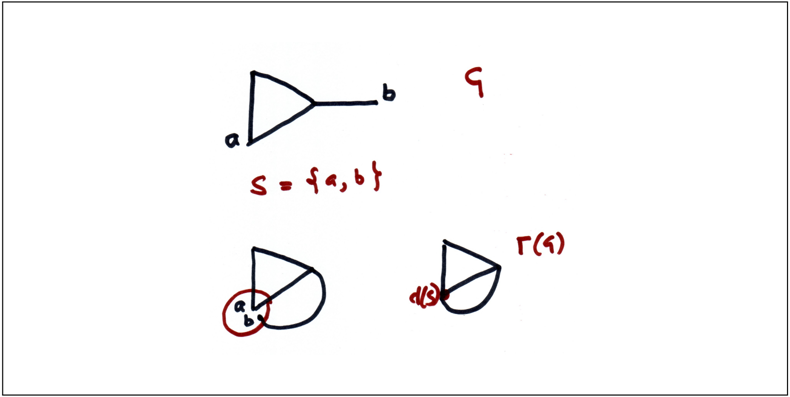


Example: Move to an unvisited vertex

Walk at current vertex $X_t \in \mathcal{B}(t)$

Given the walk selects a red config. point of X_t (if any), the probability this is paired with an config. point in $\mathcal{R}(t)$ is $\frac{r|\mathcal{R}(t)|}{|R_t|-1}$

Shrinking Vertices: First visit to a set of vertices S



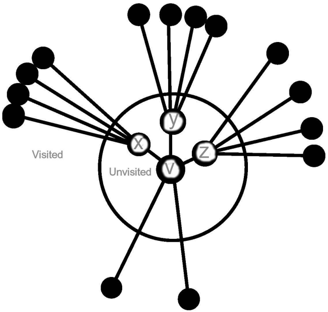
S subset of vertices of G. $\gamma(S)$ is S shrunk to a vertex
$\Gamma(G)$ is G with S shrunk to $\gamma(S)$

$$\mathbf{Pr}_G(S \text{ unvisited at step } t) \sim \mathbf{Pr}_{\Gamma(G)}(\gamma(S) \text{ unvisited at step } t)$$

Note: Notation overloaded $\Gamma(t)$ and $\Gamma(G)$–apologies

Degree of unvisited vertex

Vertex v has 3 unvisited neighbours x, y, z and 2 visited neighbours a, b, so $s = 3$, $r - s = 2$



Calculate probability that exactly $\{v, x, y, z\}$ are unvisited, and a, b visited from probability that $\{v, x, y, z\}$ are unvisited, $\{v, x, y, z, a\}$ are unvisited etc. Contract e.g. $\{v, x, y, z\}$ to a single vertex γ of degree 20 with 3 loops

The degree sequence of $\mathcal{R}(t)$

Unvisit probability

$$\mathbf{Pr}(v \in \mathcal{R}(t)) \sim e^{-t(r-2)/(r-1)n}.$$

To analyse the degree sequence of $\Gamma(t)$ we prove

Lemma

If the neighbours of v in G are $w_1, w_2, \ldots, w_r$ then

$$\mathbf{Pr}(v, w_1, \ldots, w_s \in \mathcal{R}_t, w_{s+1}, \ldots, w_r \in \mathcal{B}(t)) \sim e^{-\frac{(r-2)t}{(r-1)n}} p_t^s \,(1-p_t)^{r-s}$$

where $p_t = e^{-\frac{t(r-2)^2}{n(r-1)r}}$

We write

$$\mathbf{Pr}_{\mathcal{W}}(\{v, w_1, \ldots, w_s\} \subseteq \mathcal{R}(t) \textit{ and } \{w_{s+1}, \ldots, w_r\} \subseteq \mathcal{B}(t))$$
$$= \sum_{X \subseteq [s+1,r]} (-1)^{|X|} \mathbf{Pr}_{\mathcal{W}}((\{v, w_1, \ldots, w_s\} \cup X) \subseteq \mathcal{R}(t))$$
$$\sim \sum_{X \subseteq [s+1,r]} (-1)^{|X|} \, e^{-tp_{\gamma_X}},$$

where

$$p_{\gamma_X} \sim \frac{((r-2)(s+|X|)+r)(r-2)}{r(r-1)n}.$$

To prove this we contract $\{v, w_1, \ldots, w_s\} \cup X$ to a single vertex γ_X creating $\Gamma_X(t)$.
We then estimate the probability that γ_X hasn't been visited by a random walk on $\Gamma_X(t)$. (Unvisit probability)

For this we argue that $|\{v, w_1, \dots, w_s\} \cup X| = s + |X| + 1$

$$\pi_{\gamma_X} = \frac{r(s + |X| + 1)}{rn}$$

and

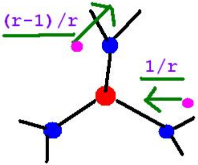
$$R_{\gamma_X} \sim \frac{(s + |X| + 1)r(r-1)}{((r-2)(s + |X|) + r)(r-2)}$$

Expression for R_{γ_X} is obtained by considering the expected number of returns to the origin in an infinite tree with branching factor $r - 1$ at each non-root vertex. At the root there are $s + |X|$ loops and $(r-2)(s + |X|) + r$ branching edges..

Reminder: R_v for random r-regular graphs

A transition on the loops returns to γ_X immediately, and a transition on any other edge is (usually) like a walk in a tree

If v is tree-like (not near any short cycles) then $R_v \sim \frac{r-1}{r-2}$



Same as: random walk on the line $(0, 1, 2, ...)$
Pr(go left) $= \frac{1}{r}$, **Pr**(go right) $= \frac{r-1}{r}$

Degree sequence of $\Gamma(t)$. Molloy-Reed

Unvisit probability

$$\mathbf{Pr}(v \in \mathcal{R}(t)) \sim e^{-t(r-2)/(r-1)n}$$

and the degree of a vertex in $\Gamma(t)$ is (approximately) binomial $Bin(r, p_t)$ where $p_t = e^{-\frac{t(r-2)^2}{n(r-1)r}}$

Once we know the degree sequence we can use the Molloy-Reed criterion to see whether or not there is a giant component. G has a giant component iff $S > 0$, where

$$S = \sum_v d_v(d_v - 2).$$

Direct calculation gives $t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2}\, n$ as the critical value

Heuristically, $t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2} n$ can be obtained from the degree sequence of unvisited vertices

Branching outward from an unvisited vertex
The probability an edge goes to another unvisited vertex:

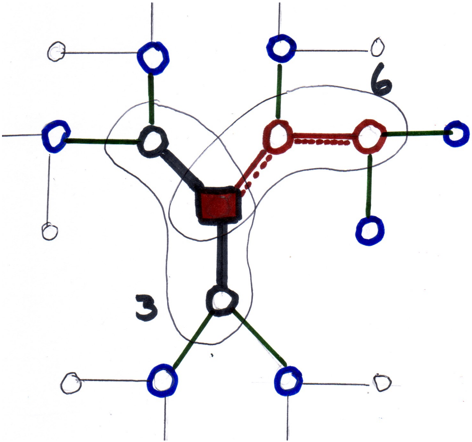
$$p_t = e^{-\frac{(r-2)^2 t}{(r-1)rn}}$$

We need branching factor $(r-1)p_t > 1$, to have a chance to get a large component

At $t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2} n$

$$\begin{aligned} (r-1)p_t &= (r-1)e^{-\frac{(r-2)^2 t}{(r-1)rn}} \\ &= (r-1)e^{-\log(r-1)} \\ &= 1 \end{aligned}$$

Rooted subtrees of the infinite regular tree



Number of rooted k-subtrees of the infinite r-regular tree

$$\frac{r}{((r-2)k+2)}\binom{(r-1)k}{k-1}$$

Number of small components in $\Gamma(t)$

$N_t = \mathbf{E}|\mathcal{R}(t)|$. Expected size of vacant set
p_t probability of a red edge
$N(k,t)$: Number of unvisited tree components of $\Gamma(t)$ with k vertices

Theorem

Let ϵ be a small positive constant. Let $1 \le k \le \epsilon \log n$ and $\epsilon n \le t \le (1-\epsilon)t_{k-1}$. Then **whp***:*

$$N(k,t) \sim \frac{r}{k((r-2)k+2)} \binom{(r-1)k}{k-1} \quad N_t\, p_t^{k-1}(1-p_t)^{k(r-2)+2}$$

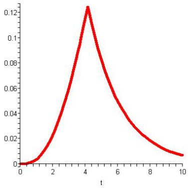
Vertices on small components of vacant set

Let

$$t^* = n \, \frac{r(r-1)}{(r-2)^2} \, \log(r-1).$$

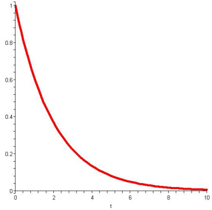
Theorem

Let $\mu(t)$ be the expected proportion of vertices on small trees. The function $\mu(t)$ increases from 0 at $t = 0$, to a maximum value $\mu^* = 1/(r-1)^{r/(r-2)}$ at $t \to t^*$, and decreases to 0 as $t \to (r-1)/(r-2)\, n \log n$

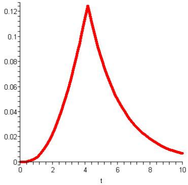


Example: $r = 3$. Vacant set as a function of $\tau = t/n$

Proportion of vertices in vacant set $N(t)/n \sim e^{-t/n((r-2)/(r-1))}$

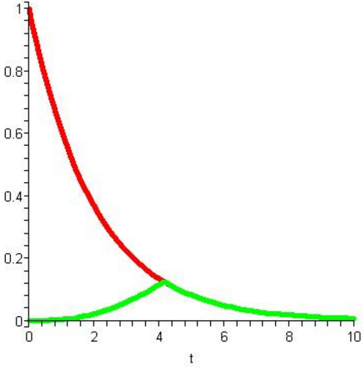


Proportion of vertices in unvisited tree components



Threshold: $r = 3$, $t^* = 6 \log 2$

$$t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2} n$$

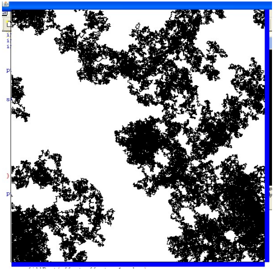


Propn. of vertices in vacant set, and on small tree components

Closing observations

- Both classes of graphs ($G(n, p)$, $G(n, r)$) exhibit threshold behavior
- The size of the giant can be estimated in the super-critical range
- The number of small components of a given size can be estimated
- The technique is simple, but seems restricted to random graphs

THANK YOU



QUESTIONS