Perimeter length of the convex hull of hyperbolic Brownian motion

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Picture

Take Brownian motion in the hyperbolic plane $\mathbb{H}^2$ (the unique complete, simply-connected two-dimensional Riemannian manifold with constant curvature -1).

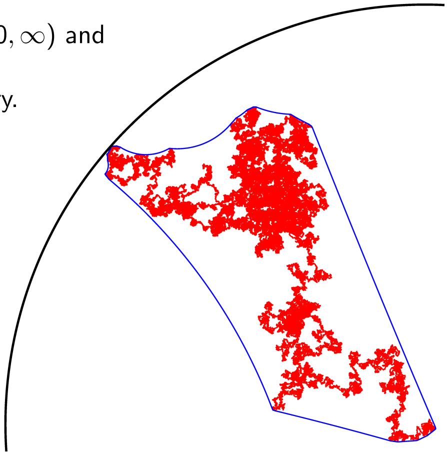
Run the process up to time $t \in (0, \infty)$ and construct the closed convex hull containing the Brownian trajectory.

Consider its perimeter length L_t (where convexity and length have the intrinsic hyperbolic sense).

The picture shows the Poincaré disk model, with $t = 10$.

Question: What can we say about expectation $\mathbb{E}\, L_t$?

E.g. as $t \to 0$ or $t \to \infty$?



Outline

Some background: Motivation

- Convex hulls of stochastic processes motivated by understanding extremal geometry of processes; seeking multidimensional extensions of one-dimensional fluctuation or record-value theory; by modelling animal territories; undertaking set estimation; etc.

- Convex hull of Euclidean Brownian motion (or random walk, Lévy process, ...) has been studied since Lévy (1955); milestones are Spitzer & Widom (1961), Letac & Takács (1980), Cranston, Hsu & March (1989), Khoshnevisan (1992), Snyder & Steele (1993), Majumdar, Comtet & Randon-Furling (2010)...

- Recent activity, e.g. Eldan (2014), W. & Xu (2015), Molchanov & Wespi (2016), McRedmond & W. (2018), Vysotsky & Zaporozhets (2018), Akopyan & Vysotsky (2021), Bang, González Cázares & Mijatović (2022), Cygan, Sandrić, Šebek & W. (2024)...

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Some background: Euclidean setting

Some comments on Euclidean Brownian motion that all fail to carry over to the hyperbolic case:

- Euclidean Brownian motion possesses scaling and time-inversion properties that mean for fixed-time statistics it suffices to take $t = 1$, and for asymptotics the $t \to \infty$ and $t \to 0$ regimes are closely related.
- Euclidean Brownian motion is angular recurrent, i.e., reaches all angles at arbitrarily large times, so the convex hull eventually fills out all space.

In Euclidean planar geometry, there's an elegant formula due to Cauchy & Crofton that expresses perimeter length of a closed convex body as the integral over all angles of the width of the body in that direction. For Brownian motion, Letac & Takács (1980) used this to compute

$$\mathbb{E}[L_t^E] = \mathbb{E} \int_0^{2\pi} \sup_{0 \le s \le t} (e_\theta^\top B_s) \,\mathrm{d}\theta = 2\pi\sqrt{t}\,\mathbb{E} \sup_{0 \le s \le 1} (e_0^\top B_s) = \sqrt{8\pi t},$$

where $B = (B_s)_{s \ge 0}$ is Brownian motion on $\mathbb{R}^2$, L_t^E is the perimeter length of the convex hull of $B[0, t]$, and $e_\theta^\top$ is projection in direction θ.

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Some background: Hyperbolic Brownian motion

> *"[Hyperbolic] Space is big. You just won't believe how vastly, hugely, mind-bogglingly big it is. I mean, you may think it's a long way down the road to the chemist's, but that's just peanuts to [hyperbolic] space."*
>
> *— [apologies to] Douglas Adams*

Describe hyperbolic BM in intrinsic polar coordinates $(R_t, \theta_t)_{t \geq s}$, for any $s > 0$, via coupled SDEs

$$R_0 := 0, \quad \mathrm{d}R_t = \frac{\mathrm{d}t}{2 \tanh R_t} + \mathrm{d}W_t^R, \text{ for all } t \in \mathbb{R}_+; \tag{1}$$

$$\Theta_s^{(s)} := 0, \quad \mathrm{d}\Theta_t^{(s)} = \frac{\mathrm{d}W_t^{(s)}}{\sinh R_t}, \text{ for all } t \geq s, \tag{2}$$

where W^R and $W^{(s)}$ are independent real-valued BMs. Then for all $t \geq s$, $\theta_t = (\theta_s + \Theta_t^{(s)})$ modulo 2π, with entrance law $\theta_s \sim \mathrm{Unif}\,[0, 2\pi)$, for every $s > 0$, inevitable thanks to the *rapid spinning* out from the origin (cf. skew-product description of transient Euclidean BM started from the origin).

Some background: Hyperbolic Brownian motion

On large scales, hyperbolic BM is very different to Euclidean Brownian motion. Indeed, hyperbolic BM is transient with a positive asymptotic speed and a (random) limiting direction:

Proposition.

It holds that $\mathbb{P}(R_t > 0 \text{ for all } t > 0) = 1$, *and, a.s.,*

$$\lim_{t\to\infty} \frac{R_t}{t} = \lim_{t\to\infty} \frac{\mathbb{E}\, R_t}{t} = \frac{1}{2}.$$

Moreover, there exists a random $\theta_\infty \sim \mathrm{Unif}\,[0, 2\pi)$, *such that*

$$\lim_{t\to\infty} \theta_t = \theta_\infty, \text{ a.s.}$$

The 'a.s.' statements here are very well known, although we could not explicitly find the statement about $\mathbb{E}\, R_t$. Proof is by direct (not hard) analysis of the SDEs (1)–(2).

Some background: Hyperbolic Brownian motion

The previous proposition gives a sense in which the hyperbolic Brownian trajectory 'converges to a line segment in a random direction'. This statement needs to be interpreted with care, however.

Comment: If we take a Euclidean Brownian motion with drift, then it satisfies a similar strong law with a limiting (in this case, non-random) direction. Euclidean scale-invariance and continuous mapping leads to a strong law for the perimeter of the convex hull, i.e., $L_t^E/t \to$ const., a.s.

In the hyperbolic case, since the hyperbolic convex hull contains the line segment, and perimeter is (still!) monotone, the above proposition does at least give a lower bound:

Corollary.

The perimeter length satisfies the "line segment" lower bounds

$$\liminf_{t\to\infty} \frac{L_t}{t} \geq 1, \text{ a.s.,} \quad \text{and} \quad \liminf_{t\to\infty} \frac{\mathbb{E}\, L_t}{t} \geq 1.$$

Outline

Expected perimeter results

Our main result expresses $\mathbb{E}\, L_t$ in terms of an *exponential functional* of Brownian motion on the line, as well as large-time and small-time asymptotics. Define

$$\mathcal{E}_t := \int_0^t \exp(2W_s - s)\mathrm{d}s, \text{ for } t \in \mathbb{R}_+,$$

where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion on $\mathbb{R}$.

Theorem (BVW 2025).
It holds that $\mathbb{E}\, L_t = \sqrt{8\pi}\, \mathbb{E}\sqrt{\mathcal{E}_t}$ *for every* $t \in \mathbb{R}_+$. *Moreover,*

$$\lim_{t \to 0} \frac{\mathbb{E}\, L_t}{\sqrt{8\pi t}} = 1, \quad \textit{and} \quad \lim_{t \to \infty} \frac{\mathbb{E}\, L_t}{2t} = 1.$$

Expected perimeter results: Remarks

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- The large-time asymptotics show that the "line segment" lower bound from the Corollary is not sharp, by a factor of 2.
- The small-time asymptotics coincide with those in the Euclidean setting, in accord with the intuition that Brownian motion experiences hyperbolic space as locally flat.
- The asymptotic results follow from the representation in terms of an exponential functional and powerful results of Hariya & Yor (2004). So in the rest of this talk, I will explain where the exponential functional comes from.

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Outline

Models of the hyperbolic plane

Recall that we described hyperbolic BM in terms of geodesic polar coordinates (R, θ) in $\mathbb{H}^2$.

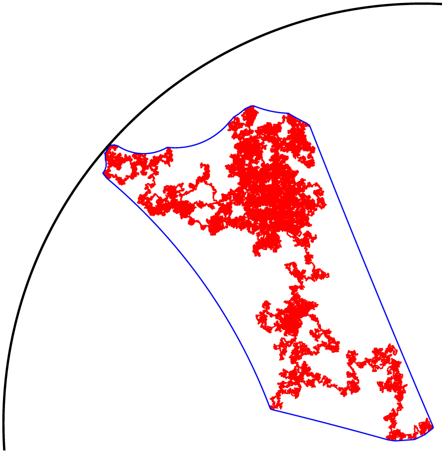
For geometrical calculations, it is very useful to represent in a model for $\mathbb{H}^2$ with a Euclidean planar domain endowed with a special metric. Three that will be useful for us:

- The Poincaré disk $\mathbb{D}_{\mathrm{P}}$. Has the advantage that it presents angular information directly, is conformal, and has relatively simple formulas.
- The Beltrami–Klein disk $\mathbb{D}_{\mathrm{K}}$. Another disk model that has the advantage that geodesics are straight lines, so convexity is easier to work with, and there is an analogue of the Cauchy formula for perimeter length that sits most naturally here.
- The Poincaré half plane $\mathbb{H}_{\mathrm{P}}$. Here the SDEs for hyperbolic BM have a nice form that reveals the exponential functional $\mathcal{E}_t$ introduced above.

Models of the hyperbolic plane: Poincaré disk

Base space is unit Euclidean disk. In polar coordinates, angle is just the geodesic polar angle, and relation between the radius in the disk and the geodesic radius is

$$r = \tanh(R/2).$$



Models of the hyperbolic plane: Beltrami–Kelin disk

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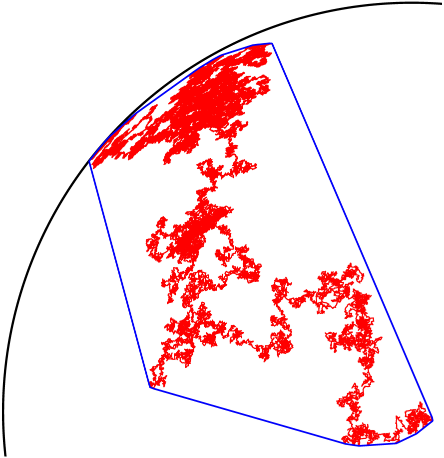
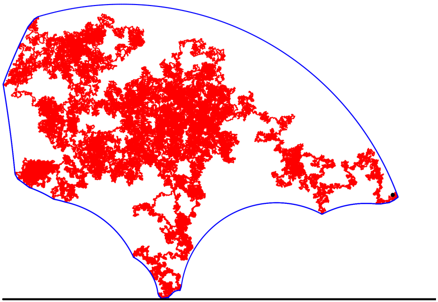


Models of the hyperbolic plane: Poincaré half plane

Base space is $\mathbb{R} \times (0, \infty)$, origin mapped to $(0, 1)$. The coordinates (x, y) are given in terms of geodesic polar coordinates (R, θ) via

$$x = \frac{\sinh R \cos\theta}{\cosh R - \sinh R \sin\theta}, \quad y = \frac{1}{\cosh R - \sinh R \sin\theta}.$$



A hyperbolic Cauchy formula

Crucial is a Cauchy-type formula from Alexander, Berg & Foote (2005) which expresses the hyperbolic perimeter length of a convex subset of the Beltrami–Klein disk $\mathbb{D}_{\mathrm{K}}$ via an integral, over all angles, of certain one-dimensional 'widths'.

To describe the formula, consider a hyperbolic convex body $K \subset \mathbb{D}_{\mathrm{K}}$. Fix an angle $\varphi \in [0, 2\pi)$ at the origin $(0,0)$ (with respect to the horizontal axis) in $\mathbb{D}_{\mathrm{K}}$.

Take point $R(\varphi) := (\cos\varphi, \sin\varphi)$ on the boundary of $\mathbb{D}_{\mathrm{K}}$, and its anticlockwise orthogonal companion $R^{\perp}(\varphi) := (-\sin\varphi, \cos\varphi)$.

Let $\ell(x, \varphi)$ denote the line through $R(\varphi)$ and $x \in K$ and let $\ell^{\perp}(\varphi)$ denote the line through $(0,0)$ and $R^{\perp}(\varphi)$, parametrized by signed Euclidean arc length as $\ell^{\perp}(\varphi) := \{\ell^{\perp}(\varphi, \lambda) : \lambda \in \mathbb{R}\}$.

For given $x \in K$ and $\varphi \in [0, 2\pi]$, let $\lambda(\varphi, x)$ denote the value of $\lambda \in \mathbb{R}$ such that $\ell^{\perp}(\varphi, \lambda) \in \ell(x, \varphi)$, i.e., the parameter corresponding to the intersection point of lines $\ell(x, \varphi)$ and $\ell^{\perp}(\varphi)$.

A hyperbolic Cauchy formula

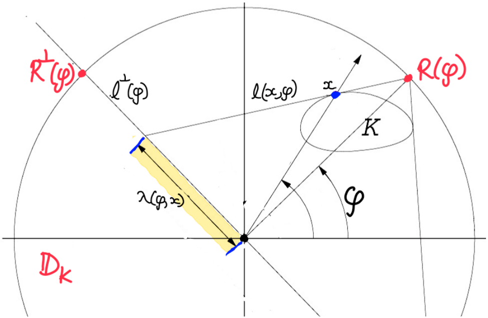


Figure adapted from Alexander, Berg & Foote (2005)

Cauchy formula of Alexander, Berg & Foote (2005) says that the hyperbolic perimeter of the $K \subset \mathbb{D}_\mathrm{K}$ is given by

$$\operatorname{perim} K = \int_0^{2\pi} \sup_{x \in K} \lambda(\varphi, x) \mathrm{d}\varphi.$$

A hyperbolic Cauchy formula

$$\operatorname{perim} K = \int_0^{2\pi} \sup_{x \in K} \lambda(\varphi, x) \mathrm{d}\varphi.$$

Since hyperbolic BM in $\mathbb{D}_{\mathrm{K}}$ is isotropic, this gives a formula for $\mathbb{E}\, L_t$ as 2π times a certain expectation of a (supremum of) a one-dimensional process. This is analogous to the Euclidean case of LETAC & TAKÁCS.

We will not go into the details here of the calculation, but it turns out that the one-dimensional process that appears is another object we have already mentioned:

Theorem.
It holds that $\mathbb{E}\, L_t = 2\pi\, \mathbb{E}\left[X_t^\star\right]$, *where* $X_t^\star := \sup_{0 \le s \le t} X_s$ *and* $X = (X_t)_{t \ge 0}$ *is the horizontal process of hyperbolic BM in the Poincaré half-plane* $\mathbb{H}_{\mathrm{P}}$.

Half-plane and exponential functionals

In $\mathbb{H}_\mathrm{P}$, the SDEs for hyperbolic BM $(X, Y) \in \mathbb{R} \times (0, \infty)$ take the nice form

$$(X_0, Y_0) = (0, 1), \quad \mathrm{d}X_t = Y_t \mathrm{d}W_t^X, \quad \mathrm{d}Y_t = Y_t \mathrm{d}W_t^Y,$$

where W^X and W^Y are independent $\mathbb{R}$-valued BMs. The SDEs can be solved explicitly to give

$$Y_t = \exp\left(W_t^Y - \frac{t}{2}\right), \quad X_t = \int_0^t Y_s \mathrm{d}W_s^X.$$

In particular, the martingale X has quadratic variation process

$$[X]_t = \int_0^t \exp\left(2W_s^Y - s\right) \mathrm{d}s \overset{d}{=} \mathcal{E}_t.$$

Then $\lim_{t\to\infty}[X]_t = [X]_\infty < \infty$, a.s., so martingale X converges (this is the limiting direction result manifest in $\mathbb{H}_\mathrm{P}$). Nevertheless, $\mathbb{E}\, X_t^\star$ turns out to grow linearly in t as $t \to \infty$.

Half-plane and exponential functionals

Lemma.
It holds that $\mathbb{E}\,X_t^\star = \sqrt{2/\pi}\,\mathbb{E}[\sqrt{\mathcal{E}_t}]$.

Proof.
Since W^X and W^Y are independent, the continuous martingale X whose quadratic variation process has the law of $\mathcal{E}$ can be represented via the time change

$$X_t = \widetilde{W}_{\mathcal{E}_t}$$

for $\widetilde{W}$ a one-dimensional BM independent of $\mathcal{E}$. So

$$X_t^\star = \sup_{0\leq s\leq t} X_s = \sup_{0\leq s\leq \mathcal{E}_t} \widetilde{W}_s \overset{d}{=} \sqrt{\mathcal{E}_t} \sup_{0\leq s\leq 1} \widetilde{W}_s.$$

Now simply use that $\mathbb{E}\sup_{0\leq s\leq 1}\widetilde{W}_s = \sqrt{2/\pi}$. □

Finishing the proof

We have shown that

- $\mathbb{E}\, L_t = 2\pi\, \mathbb{E}\, X_t^\star$ (Cauchy formula);
- $\mathbb{E}\, X_t^\star = \sqrt{2/\pi}\, \mathbb{E}\, \sqrt{\mathcal{E}_t}$ (solving the half-plane SDE).

So we get the claimed formula:

$$\mathbb{E}\, L_t = \sqrt{8\pi}\, \mathbb{E}\, \sqrt{\mathcal{E}_t}.$$

Asymptotics ($t \to \infty$) of moments of $\mathbb{E}[\mathcal{E}_t^p]$ were studied by Hariya & Yor (2004). The case $p = 1/2$ is the most delicate (for $p < 1/2$ they are uniformly bounded, for $p > 1/2$ they grow exponentially!) Using results of Hariya & Yor ($t \to 0$ can be done directly) we get

$$\lim_{t \to 0} \frac{\mathbb{E}\, L_t}{\sqrt{8\pi t}} = 1, \quad \text{and} \quad \lim_{t \to \infty} \frac{\mathbb{E}\, L_t}{2t} = 1.$$

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Outline

Concluding remarks

- For fixed t, there is (following Yor, 1992) an exact formula for $\mathbb{E}\sqrt{\mathcal{E}_t}$, but only as a (complicated) triple-integral.
- But (Yor, 2001) there is an explicit formula, involving Gamma functions, for $\mathbb{E}\sqrt{\mathcal{E}_T}$ at an independent exponential random time T. This leads to:

Corollary.

For $\lambda > 0$, let $T_\lambda \sim Exp(\lambda)$ be independent of the BM. Then

$$\mathbb{E}\, L_{T_\lambda} = G\big(\sqrt{8\lambda + 1}\big),$$

where

$$G(x) := \pi \left(\frac{x-1}{x+1}\right) \left(\frac{\Gamma\big(\frac{x-1}{4}\big)}{\Gamma\big(\frac{x+1}{4}\big)}\right)^2 .$$

Concluding remarks

- Obviously one would hope to get more understanding of L_t than just its mean. What about $\mathbb{V}\mathrm{ar}\, L_t$? Does L_t/t have a deterministic or random limit as $t \to \infty$? If so, what is it?

For details, see:

Perimeter length of the convex hull of Brownian motion in the hyperbolic plane, C. Bhattacharjee, R. Versendaal, A. Wade
ArXiv: 2502.15340

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