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Differential Geometry 12.10.2017

Homework 1-2
Starred problems due on Thursday, 26 October.

Plane curves - 1

1.1. Sketch the trace of the smooth curve given by $\boldsymbol{\alpha}(u) = (u^5, u^2 - 1)$, and mark the singular points.

1.2. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a smooth curve, and let $[a, b] \subset I$ be a closed interval. For every partition $a = u_0 < u_1 < \cdots < u_n = b$ consider the sum

$$\ell_{\boldsymbol{\alpha},P} := \sum_{i=1}^{n} \|\boldsymbol{\alpha}(u_i) - \boldsymbol{\alpha}(u_{i-1})\|$$

where P stands for the given partition. Give a geometric interpretation of $\ell_{\boldsymbol{\alpha},P}$. What length does $\ell_{\boldsymbol{\alpha},P}$ measure? Now assume that the partition becomes *finer*, i.e., $\|P\| := \max_{i=1,\ldots,n} |u_i - u_{i-1}|$ becomes smaller. What is the limit of $\ell_{\boldsymbol{\alpha},P}$ as $\|P\| \to 0$?

1.3. $(*)$ An *epicycloid* $\boldsymbol{\alpha}$ is obtained as the locus of a point on the circumference of a circle of radius r which rolls without slipping on a circle of the same radius.

(a) Sketch $\boldsymbol{\alpha}$.

(b) Show that the epicycloid can be parametrized by

$$\boldsymbol{\alpha}(u) = (2r \sin u - r \sin 2u,\ 2r \cos u - r \cos 2u), \qquad u \in \mathbb{R}.$$

Find the length of $\boldsymbol{\alpha}$ between the singular points at $u = 0$ and $u = 2\pi$.

1.4. $(*)$ (a) Let $\boldsymbol{\alpha}(u)$ and $\boldsymbol{\beta}(u)$ be two smooth plane curves. Show that

$$\frac{d}{du}(\boldsymbol{\alpha}(u) \cdot \boldsymbol{\beta}(u)) = \boldsymbol{\alpha}'(u) \cdot \boldsymbol{\beta}(u) + \boldsymbol{\alpha}(u) \cdot \boldsymbol{\beta}'(u),$$

where $\boldsymbol{\alpha}(u) \cdot \boldsymbol{\beta}(u)$ denotes a Euclidean dot product of vectors $\boldsymbol{\alpha}(u)$ and $\boldsymbol{\beta}(u)$.

Hint: write $\boldsymbol{\alpha}(u) = (\alpha_1(u), \alpha_2(u))$, $\boldsymbol{\beta}(u) = (\beta_1(u), \beta_2(u))$ and compute everything in coordinates.

(b) Let $\boldsymbol{\alpha}(u) : I \to \mathbb{R}^2$ be a smooth curve which does not pass through the origin. Suppose there exists $u_0 \in I$ such that the point $\boldsymbol{\alpha}(u_0)$ is the closest to the origin amongst all the points of the trace of $\boldsymbol{\alpha}$. Show that $\boldsymbol{\alpha}(u_0)$ is orthogonal to $\boldsymbol{\alpha}'(u_0)$.

1.5. The second derivative $\boldsymbol{\alpha}''(u)$ of a smooth plane curve $\boldsymbol{\alpha}(u)$ is identically zero. What can be said about $\boldsymbol{\alpha}$?

1.6. Let $\boldsymbol{\alpha} : (0, \pi) \to \mathbb{R}^2$ be a curve defined by

$$\boldsymbol{\alpha}(u) = (\sin u, \cos u + \log \tan \frac{u}{2})$$

The trace of $\boldsymbol{\alpha}$ is called a *tractrix*.

(a) Sketch $\boldsymbol{\alpha}$.

(b) Show that a tangent vector at $\boldsymbol{\alpha}(u_0)$ can be written as

$$\boldsymbol{\alpha}'(u_0) = (\cos u_0, -\sin u_0 + \frac{1}{\sin u_0})$$

Show that $\boldsymbol{\alpha}(u)$ is smooth, and it is regular everywhere except $u = \pi/2$.

(c) Write down the equation of a tangent line l_{u_0} to the trace of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$.

(d) Show that the distance between $\boldsymbol{\alpha}(u_0)$ and the intersection of l_{u_0} with y-axis is constantly equal to 1.

Plane curves - 2

2.1. The *catenary* is the plane curve $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ given by $\boldsymbol{\alpha}(u) = (u, \cosh u)$. It is the curve assumed by a uniform chain hanging under the action of gravity. Sketch the curve. Find its curvature.

2.2. Suppose that $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ is a regular curve, but not necessarily unit speed. Write $\boldsymbol{\alpha}(u) = (x(u), y(u))$. Find the formula for the curvature $\kappa(u)$ at the parameter value u in terms of the functions x and y (and their derivatives) at u.

Hint: consider the corresponding curve $\widetilde{\boldsymbol{\alpha}}$ parametrised by arc length. The curvature $\widetilde{\kappa}$ of $\widetilde{\boldsymbol{\alpha}}$ is then $\widetilde{\kappa}(s) = \widetilde{\boldsymbol{n}}(s) \cdot \widetilde{\boldsymbol{t}}'(s)$, where $\widetilde{\boldsymbol{t}}$ and $\widetilde{\boldsymbol{n}}$ are the unit tangent and unit normal vector of $\widetilde{\boldsymbol{\alpha}}$. Use the relation $\widetilde{\boldsymbol{\alpha}}(s) = \boldsymbol{\alpha}(\ell^{-1}(s))$, where $s = \ell(u)$ is the arc length, together with the chain rule.

2.3. $(*)$ Compute the curvature of tractrix (see Exercise 1.6) at $\boldsymbol{\alpha}(u)$.

2.4. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a smooth regular plane curve.

(a) Assume that for some $u_0 \in I$ the normal line to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$ passes through the origin. Show that for some $\epsilon > 0$ the trace $\boldsymbol{\alpha}(u_0 - \epsilon, u_0 + \epsilon)$ can be written in polar coordinates as

$$\boldsymbol{\beta}(\vartheta) = (\rho(\vartheta) \cos \vartheta, \rho(\vartheta) \sin \vartheta)$$

for an appropriate smooth function $\rho(\vartheta)$, where $\vartheta \in J$ for some interval J.

(b) Assume that all normal lines to $\boldsymbol{\alpha}$ pass through the origin. Show that the trace of $\boldsymbol{\alpha}$ is contained in a circle.

(c) Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be given in polar coordinates by

$$\boldsymbol{\alpha}(\vartheta) = (\rho(\vartheta) \cos \vartheta, \rho(\vartheta) \sin \vartheta), \qquad \vartheta \in [a, b]$$

Show that the length of $\boldsymbol{\alpha}$ is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\vartheta$$

(d) In the assumptions of (c), show that the curvature of $\boldsymbol{\alpha}$ is

$$\kappa(\vartheta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{[\rho^2 + (\rho')^2]^{3/2}}$$

2.5. Find an arc length parameter for the graphs of the following functions $f, g : (0, \infty) \to \mathbb{R}$:

(a) $f(x) = ax + b, \quad a, b \in \mathbb{R}$;

(b)$(*)$ $g(x) = \frac{8}{27} x^{3/2}$.