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Differential Geometry 15.02.2018

Homework 15-16
Starred problems due on Thursday, 1 March.

Christoffel symbols and Gauss' Theorema Egregium

15.1. Show that the Gauss curvature K of the surface of revolution locally parametrized by

$$\boldsymbol{x}(u,v) = (f(v)\cos(u), f(v)\sin(u), g(v)), \qquad (u,v) \in U,$$

(for some suitable parameter domain U) is given by

$$K = \frac{1}{2ff'}\Big(\frac{1}{1+(f'/g')^2}\Big)'.$$

If the generating curve is parametrized by arc length, show that $K = -f''/f$. Deduce Theorema Egregium in the latter case.

15.2. Let $\boldsymbol{x}\colon U \longrightarrow S$ be a parametrization of a surface S for which $E = G = 1$ and $F = \cos(uv)$ (so that uv is the angle between the coordinate curves). Determine a suitable parameter domain U on which $\boldsymbol{x}(U)$ is a surface (i.e., where the coordinate curves are not tangential). Show that

$$K = -\frac{1}{\sin(uv)}.$$

15.3. $(*)$ If the coefficients of the first fundamental form of a surface S are given by

$$E = 2 + v^2, \qquad F = 1, \qquad G = 1,$$

show that the Gauss curvature of S is given by

$$K = -\frac{1}{(1+v^2)^2}.$$

15.4. Let $\boldsymbol{x}$ be a local parametrization of a surface S such that $E = 1$, $F = 0$ and G is a function of u only. Show that

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{G_u}{2G}, \quad \Gamma^1_{22} = -\frac{G_u}{2}$$

and that all the other Christoffel symbols are zero. Hence show that the Gauss curvature K of S is given by

$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}.$$

Curves on surfaces

16.1. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be an orthonormal basis of T_pS consisting of eigenvectors of the Weingarten map $-d_p\boldsymbol{N}$ with corresponding eigenvalues κ_1, κ_2. If $\boldsymbol{e} = (\cos\vartheta)\boldsymbol{e}_1 + (\sin\vartheta)\boldsymbol{e}_2$, show, that the normal curvature κ_{n} of a curve tangential to $\boldsymbol{e}$ is given by

$$\kappa_{\mathrm{n}}(\vartheta) = \kappa_1 \cos^2\vartheta + \kappa_2 \sin^2\vartheta.$$

Deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} \kappa_{\mathrm{n}}(\vartheta)\,\mathrm{d}\vartheta = H,$$

where H denotes the mean curvature of S at p. (This justifies the term *mean curvature*).

16.2. Let $\boldsymbol{\alpha}$ be the curve defined by

$$\boldsymbol{\alpha}(t) = \varepsilon^t(\cos t, \sin t, 1) \qquad \text{for } t \in \mathbb{R}.$$

Observe that $\boldsymbol{\alpha}$ lies on the circular cone $S = \{\, (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 \,\}$.

Show that the normal curvature of $\boldsymbol{\alpha}$ in S is inversely proportional to ε^t.

16.3. Show that an asymptotic curve can only exist in the hyperbolic or flat region $\{\, p \in S \mid K(p) \le 0 \,\}$. (In other words, if a surface is elliptic everywhere, then there is no asymptotic curve.)

16.4. Let S be a surface in $\mathbb{R}^3$ with Gauss map $\boldsymbol{N}$, and let $\boldsymbol{\beta}$ be a regular curve on S not necessarily parametrized by arc length. Show that the geodesic curvature κ_{g} of $\boldsymbol{\beta}$ is given by

$$\kappa_{\mathrm{g}} = \frac{1}{\|\boldsymbol{\beta}'\|^3}(\boldsymbol{\beta}' \times \boldsymbol{\beta}'') \cdot \boldsymbol{N}.$$

16.5. Let S be Enneper's surface (see Problem 4.2) parametrized by

$$\boldsymbol{x}(u, v) = \Big(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2v, u^2 - v^2\Big), \qquad (u, v) \in \mathbb{R}^2.$$

(a) Calculate the lines of curvature.

(b) Show that the asymptotic curves are given by $u \pm v = \text{const}$.

16.6. (a) ($*$) Show that the asymptotic curves on the surface given by $x^2 + y^2 - z^2 = 1$ are straight lines.

(b) Let S be a ruled surface. What are necessary and sufficient assumptions on S for all asymptotic curves being straight lines?

Hint: use linear algebra.