

**Homework 3-4**  
**Starred problems due on Thursday, 9 November.**

## Evolute and involute

**3.1.** Let  $\alpha$  denote the catenary from Exercise 2.1. Show that

(a) the involute of  $\alpha$  starting from  $(0, 1)$  is the tractrix from Exercise 1.6 (with  $x$ - and  $y$ -axes exchanged and different parametrization);

(b) the evolute of  $\alpha$  is the curve given by

$$\beta(u) = (u - \sinh u \cosh u, 2 \cosh u)$$

(c) Find the singular points of  $\beta$  and give a sketch of its trace.

**3.2.** (\*) *Parallels.* Let  $\alpha$  be a plane curve parametrized by arc length, and let  $d$  be a real number. The curve  $\beta(u) = \alpha(u) + d\mathbf{n}(u)$  is called the *parallel* to  $\alpha$  at distance  $d$ .

(a) Show that  $\beta$  is a regular curve except for values of  $u$  for which  $d = 1/\kappa(u)$ , where  $\kappa$  is the curvature of  $\alpha$ .

(b) Show that the set of singular points of all the parallels (i.e., for all  $d \in \mathbb{R}$ ) is the evolute of  $\alpha$ .

**3.3.** Let  $\alpha(u) : I \rightarrow \mathbb{R}^2$  be a smooth regular curve. Suppose there exists  $u_0 \in I$  such that the distance  $\|\alpha(u)\|$  from the origin to the trace of  $\alpha$  is maximal at  $u_0$ . Show that the curvature  $\kappa(u_0)$  of  $\alpha$  at  $u_0$  satisfies

$$|\kappa(u_0)| \geq 1/\|\alpha(u_0)\|$$

**3.4.** *Contact with circles.* The points  $(x, y) \in \mathbb{R}^2$  of a circle are given as solutions of the equation  $C(x, y) = 0$  where

$$C(x, y) = (x - a)^2 + (y - b)^2 - \lambda$$

Let  $\alpha = (x(u), y(u))$  be a plane curve. Suppose that the point  $\alpha(u_0)$  is also on some circle defined by  $C(x, y)$ . Then  $C$  vanishes at  $(x(u_0), y(u_0))$  and the equation  $g(u) = 0$  with

$$g(u) = C(x(u), y(u)) = (x(u) - a)^2 + (y(u) - b)^2 - \lambda$$

has a solution at  $u_0$ . If  $u_0$  is a multiple solution of the equation, with  $g^{(i)}(u_0) = 0$  for  $i = 1, \dots, k-1$  but  $g^{(k)}(u_0) \neq 0$ , we say that the curve  $\alpha$  and the circle have *k-point contact* at  $\alpha(u_0)$ .

(a) Let a circle be tangent to  $\alpha$  at  $\alpha(u_0)$ . Show that  $\alpha$  and the circle have at least 2-point contact at  $\alpha(u_0)$ .

(b) Suppose that  $\kappa(u_0) \neq 0$ . Show that  $\alpha$  and the circle have at least 3-point contact at  $\alpha(u_0)$  if and only if the centre of the circle is the centre of curvature of  $\alpha$  at  $\alpha(u_0)$ .

(c) Show that  $\alpha$  and the circle have at least 4-point contact if and only if the centre of the circle is the centre of curvature of  $\alpha$  at  $\alpha(u_0)$  and  $\alpha(u_0)$  is a vertex of  $\alpha$ .

## Space curves - 1

4.1. Check that for two curves  $\alpha, \beta : I \rightarrow \mathbb{R}^3$  holds

$$(\alpha(u) \times \beta(u))' = \alpha'(u) \times \beta(u) + \alpha(u) \times \beta'(u),$$

where  $\alpha \times \beta$  is the cross-product in  $\mathbb{R}^3$ .

4.2. (\*) Find the curvature and torsion of the curve

$$\alpha(u) = (au, bu^2, cu^3).$$

4.3. (\*) Assume that  $\alpha : I \rightarrow \mathbb{R}^3$  is a regular space curve parametrized by arc length.

(a) Determine all regular curves with vanishing curvature  $\kappa$ .

*Hint:* use Theorem 4.6

(b) Show that if the torsion  $\tau$  of  $\alpha$  vanishes, then the trace of  $\alpha$  lies in a plane.

*Hint:* do NOT use Theorem 4.6

4.4. Assume that  $\alpha(s) = (x(s), y(s), 0)$ , i.e., the trace of  $\alpha$  lies in the plane  $z = 0$ . Calculate the curvature  $\kappa$  of  $\alpha$  and its torsion  $\tau$ . What is the relation of the curvature  $\kappa$  of the space curve  $\alpha$  and the (signed) curvature  $\bar{\kappa}$  of the plane curve  $\bar{\alpha} : I \rightarrow \mathbb{R}^2$  defined by  $\bar{\alpha}(s) = (x(s), y(s))$  (i.e., the projection of the space curve  $\alpha$  to the plane  $z = 0$ )?

4.5. Consider the regular curve given by

$$\alpha(s) = \left( a \cos \frac{s}{c}, a \sin \frac{s}{c}, b \frac{s}{c} \right), \quad s \in \mathbb{R},$$

where  $a, b, c > 0$  and  $c^2 = a^2 + b^2$ . The curve  $\alpha$  is called a *helix*.

(a) Show that the trace of  $\alpha$  lies on the cylinder  $x^2 + y^2 = a^2$ .

(b) Show that  $\alpha$  is parametrized by arc length.

(c) Determine the curvature and torsion of  $\alpha$  (and notice that they are both constant).

(d) Determine the equation of the plane through  $\mathbf{n}(s)$  and  $\mathbf{t}(s)$  at each point of  $\alpha$  (this plane is called the *osculating plane*).

(e) Show that the line through  $\alpha(s)$  in direction  $\bar{\mathbf{n}}(s)$  meets the axis of the cylinder orthogonally.

(f) Show that the tangent lines to  $\alpha$  make a constant angle with the axis of the cylinder.

*Remark:* In fact, a helix can be characterized by (a) and (f). If we drop (a), then we obtain a *generalized helix* (see next homework). Another way how to characterize a helix is by (c), i.e., the fact that the curvature and torsion are constant. *Why?*