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Questions from Problems classes

Problems Class 1 (2 November 2017)

Plane curves

1. Consider a curve $\boldsymbol{\alpha}(u) = r(u)(\cos u, \sin u)$, where $r : (0, \infty) \to (0, \infty)$ is a function.
   (a) Show that $\boldsymbol{\alpha}$ is a regular curve.
   (b) Let $r(u) = \frac{1}{1+u}$. Is the length of $\boldsymbol{\alpha}$ finite?
   (c) Show that $\kappa(u) = \frac{2r'^2 - rr' + r^2}{(r^2 + r'^2)^{3/2}}$ (see also HW 2.4).
   For the case of $r(u) = u$, show that $\kappa(u) \to 0$ as $u \to \infty$.
   For the case of $r(u) = \frac{1}{1+u}$ show that $\kappa(u) \to \infty$ as $u \to \infty$.
2. Let $\boldsymbol{\alpha}(u) = (u, \frac{1}{u})$ be a hyperbola considered on the domain $\{u > 0\}$.
   (a) Find the curvature of $\boldsymbol{\alpha}$.
   (b) Are there inflection points and vertices?
   (c) Find the evolute of $\boldsymbol{\alpha}$. Find the singular points of the evolute.
3. Compute the involute of the circle $(\cos s, \sin s)$. Check that the circle is the evolute for this curve.

Problems Class 2 (16 November 2017)

Space curves and surfaces

1. Given the curve $\boldsymbol{\alpha}(u) = (2\cos u + 2\sin u, 2\sin u - 2\cos u, u)$,
   (a) find the length between $p = \boldsymbol{\alpha}(u_0)$ and $u = \boldsymbol{\alpha}(u_1)$.
   (b) Show that $\boldsymbol{\alpha}$ is a *generalised helix*, i.e. the tangents to $\boldsymbol{\alpha}$ make a constant angle with some fixed direction.
   (c) Compute κ and τ.
2. Given
$$\boldsymbol{\alpha}(u) = \begin{cases} (u, 0, e^{-1/u^2}), & \text{if } u > 0, \\ (u, e^{-1/u^2}, 0), & \text{if } u < 0, \\ (0,0,0), & \text{if } u = 0, \end{cases}$$
   show that
   (a) $\boldsymbol{\alpha}$ is a smooth regular curve.
   (b) $\kappa(u) \neq 0$ if $u \neq 0$ and $u \neq \pm\sqrt{\frac{2}{3}}$.
   (c) Find the osculating plane at $\boldsymbol{\alpha}(u)$.
3. $S = \{u^3 - u, u^2 - 1, v\}$. Is it a local parametrisation?
4. $f(x, y, z) = xyz$. For which $c \in \mathbb{R}$ is $f(x, y, z) = c$ a regular surface?

Problems Class 3 (30 November 2017)

Special surfaces. First fundamental form.

1. Let $\boldsymbol{\alpha}(v) = (\frac{1}{v}, 0, \operatorname{arccosh} v - \frac{\sqrt{v^2-1}}{v})$, $v \in (1, \infty)$ be a curve, and let S be a surface of revolution defined by the curve $\boldsymbol{\alpha}$:

$$x(u,v) = (\frac{\cos u}{v}, \frac{\sin u}{v}, \operatorname{arccosh} v - \frac{\sqrt{v^2-1}}{v}).$$

   Show that the first fundamental form of S coincides with the one of the upper half-plane model of the hyperbolic plane (see Example 7.16), i.e. $E = G = 1/v^2$, $F = 0$.

2. In the upper half plane $\mathbb{H}$ (see Example 7.16), consider the curve $\boldsymbol{\alpha}_r(\theta) : (\theta_0, \theta_1) \to \mathbb{H}$ given by $\boldsymbol{\alpha}_r(\theta) = (r\cos\theta, r\sin\theta)$ where $r > 0$.
   (a) What do you think without computations, which one is longer, $\boldsymbol{\alpha}_1$ or $\boldsymbol{\alpha}_{100}$?
   (b) Compute the length of the curve $\boldsymbol{\alpha}_r$.

3. Consider a canal surface parametrised by

$$x(u,v) = \boldsymbol{\alpha}(u) + \boldsymbol{n}(u)\cos v + \boldsymbol{b}(u)\sin v$$

   where $\boldsymbol{\alpha}(u) = (2\cos\frac{u}{2}, 2\sin\frac{u}{2}, 0)$. Show that $x(u,v)$ is a surface of revolution.

Problems Class 4 (14 December 2017)

Gauss map. Coordinate curves.

1. Compute Gauss map of the catenoid $x(u,v) = (\cosh v\cos u, \cosh v \sin u, v)$.
2. Let $f(x,y,z) = (x^2+y^2-1)^2$. For which $c \in \mathbb{R}$ the set $\{f(x,y,z) = c\}$ is a regular surface?
3. Let $x(u,v) = (u, v, \sqrt{1-u^2-v^2})$ be a hemisphere.
   (a) Find E, F, G.
   (b) Compute the angle between coordinate curves at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$.
   (c) Let $\alpha(s) = x(\frac{1}{2}\cos s, \frac{1}{2}\sin s)$, $s \in (0, 2\pi)$. Find the length $l(\alpha)$.

Problems Class 5 (1 February 2018)

Isometries of hyperbolic plane.

1. Let $\mathbb{H} = (u, v) \in \mathbb{R}^2 \mid v > 0$ be the upper half-plane with the second fundamental form given by $E = \frac{1}{v^2}, F = 0, G = \frac{1}{v^2}$. Let $f : \mathbb{H} \to H$ be given by $f(z) = az + b$, where $a, b \in \mathbb{R}$, $a > 0$. Show that f is an isometry of $\mathbb{H}$.

2. The same question for $f = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$.

Principal directions in terms of I_p and II_p.

3. Let $\{x_u, x_v\}$ be a basis of T_pS, $w = \lambda x_u + \mu x_v$.

   Show that w is a principal direction if and only if $\det \begin{pmatrix} \mu^2 & -\lambda\mu & \lambda^2 \\ E & F & G \\ L & M & N \end{pmatrix} = 0$

Problems Class 6 (15 January 2018)

Computation of Gauss and mean curvatures. Minimal surfaces.

1. Consider a surfaces S given by the parametrisation $x(r, \varphi) = (\frac{r}{\sqrt{2}} \cos(\sqrt{2}\varphi), \frac{r}{\sqrt{2}} \sin(\sqrt{2}\varphi), r)$, $r \in (0, \infty), \varphi \in \mathbb{R}$.

   (a) Find Gauss and mean curvatures of S.

   (b) Show that S is locally isometric to a plane. Notice that the mean curvature of S is not zero (unlike the mean curvature of the plane).

   (c) Is S a minimal surface?

2. Show that the helicoid $(av \cos u, av \sin u, bu)$, $u \in \mathbb{R}$, $v \in (0, \infty)$ is a minimal surface.

Remark. A helicoid is the only ruled surface (other than the plane) which is also a ruled surface (see Do Carmo, Section 3.5 B, Example 6.).

Problems Class 7 (1 March 2018)
Curves on surfaces

1. Find the asymptotic curves on $x^2 + y^2 - z^2 = 1$ (computation-free solution).
2. Let κ_1 and κ_2 be principal curvatures at $p \in S$. Show that κ_1, κ_2 are the minimal and the maximal value of the normal curvature $\kappa_n(\boldsymbol{w})$, where $||\boldsymbol{w}|| = 1$, $w \in T_pS$.
3. Let S be the elliptic paraboloid $x = (u, v, u^2 + v^2)$. Find the lines of curvature on S.

Problems Class 7 (15 March 2018)
Gauss-Bonnet Theorems

1. The aim of this question is to verify the Gauss-Bonnet theorem for a region R on the surface S given by the local parametrisation $x(u, v) = (v \cos u, v \sin u, v^2)$, where the region R is defined by $0 \leq u \leq 2\pi$, $0 \leq v < 1$.

   (a) State the global Gauss-Bonnet Theorem.

   (b) Compute the coefficients of the first and second fundamental forms on S.

   (c) Compute Gauss curvature K, calculate $\int_R K dA$.

   (d) Show that the curve $\gamma(u) = x(u, 1)$ is unit speed. Find the geodesic curvature κ_g and compute $\int_{\partial R} \kappa_g ds$.

   (e) Compute the Euler characteristic $\chi(R)$ of the region R. Verify the Gauss-Bonnet theorem for the region R.

   Remark: in part (d), be careful to chose the correct orientation of γ. By definition, it should be counter-clockwise – when the normal to the surface is upward. Notice that the opposite choice of the orientation will give the opposite sign to the $\int_{\partial R} \kappa_g ds$ and the computation in part (e) will not work!

2. We have also (very briefly) discussed oriented closed surfaces of constant curvature:

   It is known that an oriented closed surface is a sphere with a non-negative integer number of handles (a sphere with g handles is called a surface of genus g and will be denoted by S_g).

   (a) (not discussed in the Problems Class) Show by induction that $\chi(S_g) = 2 - 2g$.

   (b) Suppose that S_g is a surface of constant curvature K. Show that

   - if $g = 0$ then $K > 0$;
   - if $g = 1$ then $K = 0$;
   - if $g > 1$ then $K < 0$.