Durham University
Anna Felikson

Solutions 1-2

1.3. ($*$) An *epicycloid* $\boldsymbol{\alpha}$ is obtained as the locus of a point on the circumference of a circle of radius r which rolls without slipping on a circle of the same radius.

(a) Sketch $\boldsymbol{\alpha}$.

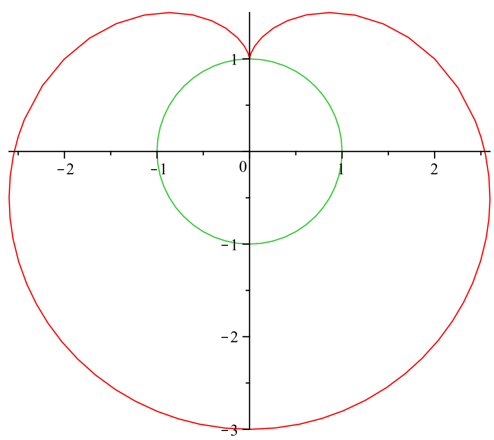
(b) Show that the epicycloid can be parametrized by

$$\boldsymbol{\alpha}(u) = (2r\sin u - r\sin 2u,\ 2r\cos u - r\cos 2u), \qquad u \in \mathbb{R}.$$

Find the length of $\boldsymbol{\alpha}$ between the singular points at $u = 0$ and $u = 2\pi$.

Solution:

The graph of the epicycloid is illustrated below for the value $r = 1$.



The inner (green) circle centered at $(0, 0)$ is fixed, and the second circle C rotates around it with a marked point on its perimeter tracing out the epicycloid. This point is at the bottom of the rotating circle at the moment $u = 0$ when the rotating circle is just on top of the fixed circle, i.e., at position $(0, r)$. As u increases, the center of C moves clockwise around the origin, and so does the point of contact between the fixed and the rotating circle, and also so does the marked point around the center of C in relation to the point of contact.

At the time u the center of the rotating circle C is located at $(2r\sin u, 2r\cos u)$. To this moment C has rotated clockwise around its moving center by a total length of $2ru$, where u is measured in radians. Therefore, the point of contact between the two circles, seen from the moving center of C, has moved clockwise by the angle u around its moving center, and the position of the point of contact relative to this moving center is $(r\sin(\pi + u), r\cos(\pi + u))$. The marked point has moved clockwise away from the point of contact by the same angle, and is therefore at position $(r\sin(\pi + 2u), r\cos(\pi + 2u))$ relative to the center of the moving circle. This means that the marked point lies at

$$(2r\sin u, 2r\cos u) + (r\sin(\pi + 2u), r\cos(\pi + 2u)) = (2r\sin u - r\sin 2u, 2r\cos u - r\cos 2u).$$

Now let

$$\boldsymbol{\alpha}(u) = (2r\sin u - r\sin 2u, 2r\cos u - r\cos 2u).$$

Then

$$\begin{aligned}
\boldsymbol{\alpha}'(u) &= 2r(\cos u - \cos 2u, -(\sin u - \sin 2u)) \\
\|\boldsymbol{\alpha}'(u)\|^2 &= 4r^2(2 - 2(\cos(-u)\cos(2u) - \sin(-u)\sin(2u)) \\
&= 4r^2(2 - 2\cos(2u-u)) = 4r^2(2 - 2\cos u) \\
&= 4r^2(2 - 2(\cos(u/2)\cos(u/2) - \sin(u/2)\sin(u/2))) \\
&= 16r^2\sin^2(u/2).
\end{aligned}$$

This implies that $\|\boldsymbol{\alpha}'(u)\| = 4r\sin(u/2)$ and

$$l(\boldsymbol{\alpha}) = \int_0^{2\pi} \|\boldsymbol{\alpha}'(u)\| du = 4r\int_0^{2\pi} \sin\frac{u}{2} du = 4r\left(-2\cos\frac{u}{2}\Big|_0^{2\pi}\right) = -8r(\cos\pi - \cos 0) = 16r.$$

1.4. $(*)$ (a) Let $\boldsymbol{\alpha}(u)$ and $\boldsymbol{\beta}(u)$ be two smooth plane curves. Show that

$$\frac{d}{du}(\boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}(u)) = \boldsymbol{\alpha}'(u)\cdot\boldsymbol{\beta}(u) + \boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}'(u),$$

where $\boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}(u)$ denotes a Euclidean dot product of vectors $\boldsymbol{\alpha}(u)$ and $\boldsymbol{\beta}(u)$.

(b) Let $\boldsymbol{\alpha}(u) : I \to \mathbb{R}^2$ be a smooth curve which does not pass through the origin. Suppose there exists $u_0 \in I$ such that the point $\boldsymbol{\alpha}(u_0)$ is the closest to the origin amongst all the points of the trace of $\boldsymbol{\alpha}$. Show that $\boldsymbol{\alpha}(u_0)$ is orthogonal to $\boldsymbol{\alpha}'(u_0)$.

Solution:

(a) Let $\boldsymbol{\alpha}(u) = (\alpha_1(u), \alpha_2(u))$, $\boldsymbol{\beta}(u) = (\beta_1(u), \beta_2(u))$. Then

$$\boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}(u) = \alpha_1(u)\beta_1(u) + \alpha_2(u)\beta_2(u)$$

Thus,

$$\begin{aligned}
\frac{d}{du}(\boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}(u)) &= \frac{d}{du}(\alpha_1(u)\beta_1(u) + \alpha_2(u)\beta_2(u)) = \alpha_1'(u)\beta_1(u) + \alpha_1(u)\beta_1'(u) + \alpha_2'(u)\beta_2(u) + \alpha_2(u)\beta_2'(u) = \\
&= (\alpha_1'(u)\beta_1(u) + \alpha_2'(u)\beta_2(u)) + (\alpha_1(u)\beta_1'(u) + \alpha_2(u)\beta_2'(u)) = \boldsymbol{\alpha}'(u)\cdot\boldsymbol{\beta}(u) + \boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}'(u)
\end{aligned}$$

(b) Since the point $\boldsymbol{\alpha}(u_0)$ is the closest to the origin, the derivative of the function $\|\boldsymbol{\alpha}(u)\|^2$ vanishes at point u_0. Using the equality $\|\boldsymbol{\alpha}(u)\|^2 = \boldsymbol{\alpha}(u)\cdot\boldsymbol{\alpha}(u)$ and (a), we obtain

$$0 = \frac{d}{du}\|\boldsymbol{\alpha}(u)\|^2|_{u_0} = \frac{d}{du}\boldsymbol{\alpha}(u)\cdot\boldsymbol{\alpha}(u) = 2\boldsymbol{\alpha}'(u_0)\cdot\boldsymbol{\alpha}(u_0),$$

so $\alpha'(u_0)$ and $\alpha(u_0)$ are orthogonal.

1.5. The second derivative $\boldsymbol{\alpha}''(u)$ of a smooth plane curve $\boldsymbol{\alpha}(u)$ is identically zero. What can be said about $\boldsymbol{\alpha}$?

Solution: Since $\boldsymbol{\alpha}''(u) \equiv 0$, the tangent vector $\boldsymbol{\alpha}'(u)$ is constant, which implies that $\boldsymbol{\alpha}(u)$ is either a constant speed parametrization of a line or just a single point.

1.6. Let $\boldsymbol{\alpha} : (0, \pi) \to \mathbb{R}^2$ be a curve defined by

$$\boldsymbol{\alpha}(u) = (\sin u, \cos u + \log \tan \frac{u}{2})$$

The trace of $\boldsymbol{\alpha}$ is called a *tractrix.*

(a) Sketch $\boldsymbol{\alpha}$.

(b) Show that a tangent vector at $\boldsymbol{\alpha}(u_0)$ can be written as

$$\boldsymbol{\alpha}'(u_0) = (\cos u_0, -\sin u_0 + \frac{1}{\sin u_0})$$

Show that $\alpha(u)$ is smooth, and it is regular everywhere except $u = \pi/2$.

(c) Write down the equation of a tangent line l_{u_0} to the trace of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$.

(d) Show that the distance between $\boldsymbol{\alpha}(u_0)$ and the intersection of l_{u_0} with y-axis is constantly equal to 1.

Solution: The equation of a tangent line l_{u_0} to the trace of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$ can be written as $r(v) = \boldsymbol{\alpha}(u_0) + v\boldsymbol{\alpha}'(u_0)$, or

$$r(v) = (\sin u_0 + v \cos u_0, \cos u_0 + \log \tan \frac{u_0}{2} - v \sin u_0 + v \frac{1}{\sin u_0})$$

The square of the distance between $r(v)$ and $\boldsymbol{\alpha}(u_0)$ is equal to $v^2 \|\boldsymbol{\alpha}'(u_0)\|^2$. The line intersects y-axis at $v = -\tan u_0$, so (the square of) the required distance is equal to

$$\tan^2 u_0 \|(\cos u_0, -\sin u_0 + \frac{1}{\sin u_0})\|^2 = \tan^2 u_0 (\cos^2 u_0 + \sin^2 u_0 - 2 + \frac{1}{\sin^2 u_0}) = \tan^2 u_0 (\frac{1}{\sin^2 u_0} - 1) = 1$$

2.1. The *catenary* is the plane curve $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ given by $\boldsymbol{\alpha}(u) = (u, \cosh u)$. It is the curve assumed by a uniform chain hanging under the action of gravity. Sketch the curve. Find its curvature.

Solution:

Since $\boldsymbol{\alpha}(u) = (u, \cosh u)$, we can write

$$\boldsymbol{\alpha}'(u) = (1, \sinh u),$$

so that

$$\|\alpha'(u)\| = \sqrt{1 + \sinh^2 u} = \cosh u$$

and

$$\boldsymbol{\alpha}''(u) = (0, \cosh u)$$

Now,

$$\kappa(u) = \frac{x'(u)y''(u) - x''(u)y'(u)}{\|\alpha'(u)\|^3} = \frac{\cosh u}{\cosh^3 u} = \frac{1}{\cosh^2 u}$$

2.2. Suppose that $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ is a regular curve, but not necessarily unit speed. Write $\boldsymbol{\alpha}(u) = (x(u), y(u))$. Find the formula for the curvature $\kappa(u)$ at the parameter value u in terms of the functions x and y (and their derivatives) at u.

Solution:

We can write the unit tangent vector as

$$\boldsymbol{t}(u) = \frac{\boldsymbol{\alpha}'(u)}{\|\alpha'(u)\|} = \frac{1}{\|\alpha'(u)\|}(x'(u), y'(u)),$$

so the unit normal vector can be written as

$$\boldsymbol{n}(u) = \frac{1}{\|\alpha'(u)\|}(-y'(u), x'(u))$$

To compute the curvature $\kappa(u)$ we need to compute the vector $\boldsymbol{t}'(s)|_u$, where s is an arc length parameter and $s = l(u)$ for l to be the lenght function. By the chain rule, we have

$$\boldsymbol{t}'(s)|_u = \frac{d\boldsymbol{t}}{du}\frac{du}{ds},$$

where

$$\frac{du}{ds} = (l^{-1})'(s) = \frac{1}{l'(u)} = \frac{1}{\|\alpha'(u)\|}.$$

Thus,

$$\boldsymbol{t}'(s)|_u = \frac{1}{\|\alpha'(u)\|}\frac{d}{du}\left(\frac{(x'(u), y'(u))}{\|\alpha'(u)\|}\right) = \frac{1}{\|\alpha'(u)\|}\frac{d}{du}\left(\frac{(x'(u), y'(u))}{(x'(u)^2 + y'(u)^2)^{1/2}}\right) = \frac{x'(u)y''(u) - x''(u)y'(u)}{(x'(u)^2 + y'(u)^2)^2}(-y'(u), x'(u))$$

(some work is required to obtain the last equality above...)

Therefore,

$$\kappa(u) = \boldsymbol{n}(u) \cdot \boldsymbol{t}'(s)|_u = \frac{1}{\|\alpha'(u)\|}\frac{x'(u)y''(u)) - x''(u)y'(u)}{(x'(u)^2 + y'(u)^2)^2}\|(-y'(u), x'(u))\|^2 = \frac{x'(u)y''(u) - x''(u)y'(u)}{(x'(u)^2 + y'(u)^2)^{3/2}}$$

2.3. ($*$) Compute the curvature of tractrix (see Exercise 1.6) at $\boldsymbol{\alpha}(u)$.

Solution:

Using the formula above and the expressions for $\boldsymbol{\alpha}'(u)$ and $\boldsymbol{\alpha}''(u)$

$$\boldsymbol{\alpha}'(u) = (\cos u, -\sin u + \frac{1}{\sin u}) \qquad \text{and} \qquad \boldsymbol{\alpha}''(u) = (-\sin u, -\cos u - \frac{\cos u}{\sin^2 u})$$

we compute

$$\kappa(u) = \frac{\cos u(-\cos u - \frac{\cos u}{\sin^2 u}) - (-\sin u)(-\sin u + \frac{1}{\sin u})}{(\cos^2 u + (-\sin u + \frac{1}{\sin u})^2)^{3/2}} = \frac{-\cos^2 u(1 + \frac{1}{\sin^2 u}) - (\sin^2 u - 1)}{(\cos^2 u + \sin^2 u - 2 + \frac{1}{\sin^2 u})^{3/2}} =$$

$$= \frac{-\cos^2 u - \frac{\cos^2 u}{\sin^2 u} - (-\cos^2 u)}{(\frac{1}{\sin^2 u} - 1)^{3/2}} = \frac{-\frac{\cos^2 u}{\sin^2 u}}{(\frac{\cos^2 u}{\sin^2 u})^{3/2}} = -|\tan u|$$

2.4. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a smooth regular plane curve.

(a) Assume that for some $u_0 \in I$ the normal line to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$ passes through the origin. Show that for some $\epsilon > 0$ the trace $\boldsymbol{\alpha}(u_0 - \epsilon, u_0 + \epsilon)$ can be written in polar coordinates as

$$\boldsymbol{\beta}(\vartheta) = (\rho(\vartheta)\cos\vartheta, \rho(\vartheta)\sin\vartheta)$$

for an appropriate smooth function $\rho(\vartheta)$, where $\vartheta \in J$ for some interval J.

(b) Assume that all normal lines to $\boldsymbol{\alpha}$ pass through the origin. Show that the trace of $\boldsymbol{\alpha}$ is contained in a circle.

(c) Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be given in polar coordinates by

$$\boldsymbol{\alpha}(\vartheta) = (\rho(\vartheta)\cos\vartheta, \rho(\vartheta)\sin\vartheta), \qquad \vartheta \in [a, b]$$

Show that the length of $\boldsymbol{\alpha}$ is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\vartheta$$

(d) In the assumptions of (c), show that the curvature of $\boldsymbol{\alpha}$ is

$$\kappa(\vartheta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{[\rho^2 + (\rho')^2]^{3/2}}$$

Solution:

(a) Since the normal line at $\boldsymbol{\alpha}(u_0)$ passes through the origin, the tangent vector $\boldsymbol{\alpha}'(u_0)$ is orthogonal to the vector $\boldsymbol{\alpha}(u_0)$. Write $\boldsymbol{\alpha}(u) = (x(u), y(u))$, and without loss of generality assume that $x'(u_0) \neq 0$ (otherwise rotate the whole picture around the origin by a small angle). By the latter assumption, we have $y'(u_0)/x'(u_0) \neq \infty$ (geometrically, $y'(u_0)/x'(u_0)$ is the tangent of the angle $\varphi(u_0)$ forming by the tangent vector $\boldsymbol{\alpha}'(u_0)$ and the x-axis).

By smoothness of $\boldsymbol{\alpha}$, we can choose a small ϵ such that for every $u \in (u_0 - \epsilon, u_0 + \epsilon)$ the angle $\varphi(u)$ forming by the tangent vector $\boldsymbol{\alpha}'(u)$ and the x-axis differs from $\varphi(u_0)$ not too much (say, by $\pi/100$ at most). This implies that for any $u \in (u_0-\epsilon, u_0+\epsilon)$ the line passing through the origin and $\boldsymbol{\alpha}(u)$ intersects $\boldsymbol{\alpha}(u_0-\epsilon, u_0+\epsilon)$ at $\boldsymbol{\alpha}(u)$ only.

Now, taking $\vartheta = \varphi(u) - \frac{\pi}{2}$ and $\rho(\vartheta) = \|\boldsymbol{\alpha}(u)\|$ (draw the picture!!!) we obtain the required parametrization.

(b) Take any $u_0 \in I$ and, as in (a), parametrize $\boldsymbol{\alpha}$ in some neighborhood of $\boldsymbol{\alpha}(u_0)$ by

$$\boldsymbol{\beta}(\vartheta) = \boldsymbol{\alpha}(u(\vartheta)) = (\rho(\vartheta)\cos\vartheta, \rho(\vartheta)\sin\vartheta)$$

Now

$$\boldsymbol{\beta}'(\vartheta) = (\rho'(\vartheta)\cos\vartheta - \rho(\vartheta)\sin\vartheta, \rho'(\vartheta)\sin\vartheta + \rho(\vartheta)\cos\vartheta)$$

By assumptions, $\boldsymbol{\beta}'(\vartheta)$ is orthogonal to $\boldsymbol{\beta}(\vartheta)$, so

$$0 = \boldsymbol{\beta}'(\vartheta)\cdot\boldsymbol{\beta}(\vartheta) = (\rho'(\vartheta)\cos\vartheta - \rho(\vartheta)\sin\vartheta)\rho(\vartheta)\cos\vartheta + (\rho'(\vartheta)\sin\vartheta + \rho(\vartheta)\cos\vartheta)\rho(\vartheta)\sin\vartheta = \rho'\rho,$$

which implies that $\rho' \equiv 0$. Therefore, $\rho(\vartheta) = r$ is constant in some neighborhood of every $u \in I$, so it is constant on I (prove this implication!). Thus, the trace of $\boldsymbol{\beta}$ (which coincides with the trace of $\boldsymbol{\alpha}$) is contained in a circle of radius r centered at the origin.

(c) By definition,

$$l(\boldsymbol{\alpha}) = \int_a^b \|\boldsymbol{\alpha}'(\vartheta)\|\,d\vartheta = \int_a^b \sqrt{(\rho'(\vartheta)\cos\vartheta - \rho(\vartheta)\sin\vartheta)^2 + (\rho'(\vartheta)\sin\vartheta + \rho(\vartheta)\cos\vartheta^2)}\,d\vartheta =$$

$$= \int_a^b \sqrt{\rho'(\vartheta)^2(\cos^2\vartheta + \sin^2\vartheta) + \rho'(\vartheta)\rho(\vartheta)(-2\cos\vartheta\sin\vartheta + 2\cos\vartheta\sin\vartheta) + \rho(\vartheta)^2(\sin^2\vartheta + \cos^2\vartheta)}\,d\vartheta =$$

$$= \int_a^b \sqrt{\rho^2 + (\rho')^2}\,d\vartheta$$

(d) Apply the formula for the curvature from Exercise 2.2 and the expression for $\boldsymbol{\alpha}'(\vartheta)$ from (c).

2.5. Find an arc length parameter for the graphs of the following functions $f, g : (0,\infty) \to \mathbb{R}$:

(a) $f(x) = ax + b, \quad a, b \in \mathbb{R}$;

(b)(*) $g(x) = \frac{8}{27}x^{3/2}$.

Solution:

Parametrize the curves by $\boldsymbol{\alpha}(x) = (x, f(x))$ and $\boldsymbol{\beta}(x) = (x, g(x))$, and choose $x_0 = 0$.

(a) By definition,

$$s = l(x) = \int_0^x \|\alpha'(u)\|\,du = \int_0^x \|(1, f'(u))\|\,du = \int_0^x \sqrt{1+a^2}\,du = x\sqrt{1+a^2}$$

Thus,

$$x = \frac{s}{\sqrt{1+a^2}},$$

and the curve

$$\tilde{\boldsymbol{\alpha}}(s) = \left(\frac{s}{\sqrt{1+a^2}}, \frac{as}{\sqrt{1+a^2}} + b\right)$$

is an arc length parametrization of the graph of $f(x)$.

(b) Similar to (a), we write

$$s = l(x) = \int_0^x \|\beta'(u)\|\,du = \int_0^x \|(1, \frac{4}{9}\sqrt{u})\|\,du = \int_0^x \sqrt{1 + \frac{16}{81}u}\,du = \frac{81}{16}\frac{2}{3}(1+\frac{16}{81}u)^{3/2}|_0^x = \frac{27}{8}((1+\frac{16}{81}x)^{3/2}-1),$$

which implies

$$x = \frac{81}{16}\left((\frac{8}{27}s+1)^{2/3} - 1\right)$$