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Differential Geometry 1.2.2020

Solutions 11-12

11.1. Let $a > 0$. Construct explicitly a local isometry from the plane $P = \{ (u, v, 0) \in \mathbb{R}^3 \mid u, v \in \mathbb{R} \}$ onto the cylinder $S = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2 \}$.

Solution:

A canonical parametrization of the plane P is

$$\boldsymbol{x} \colon U = \mathbb{R}^2 \longrightarrow P, \qquad \boldsymbol{x}(u, v) = (u, v, 0).$$

Clearly, $\boldsymbol{x}_u = (1, 0, 0)$, $\boldsymbol{x}_v = (0, 1, 0)$ and $E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle = 1$, $F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = 0$ and $G = \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle = 1$.

We define a candidate for the isometry via this parametrisation

$$f \colon P \longrightarrow S, \qquad f(u, v, 0) := (a \cos(\omega u), a \sin(\omega u), v)$$

for some positive constant $\omega > 0$ (we could also interchange the role of u and v) (more precisely, we define $f \circ \boldsymbol{x} \colon U \longrightarrow S$). In order to check that f is a local isometry, we just need to calculate the coefficents of the fundamental form of S with respect to the (local) parametrisation $f \circ \boldsymbol{x}$, and see whether they equal E, F and G. But here we have

$$f_u = (f \circ \boldsymbol{x})_u = (-a\omega \sin(\omega u), a\omega \cos(\omega u), 0) \quad \text{and} \quad f_v = (f \circ \boldsymbol{x})_v = (0, 0, 1),$$

so that

$$\widetilde{E} = \langle f_u, f_u \rangle = a^2\omega^2, \quad \widetilde{F} = \langle f_u, f_v \rangle = 0 \quad \text{and} \quad \widetilde{G} = \langle f_v, f_v \rangle = 1.$$

We have $\widetilde{F} = F$ and $\widetilde{G} = G$. In order to have $\widetilde{E} = E$, we need $\omega = 1/a$, then f is a local isometry (by Proposition 8.15).

11.2. $(*)$ Let b be a positive number such that $\sqrt{1 + b^2}$ is an integer n. Let S be the circular cone obtained by rotating the curve given by $\boldsymbol{\alpha}(v) = (v, 0, bv)$, $v > 0$, about the z-axis. Let the coordinate xy-plane P be parametrized by polar coordinates (r, ϑ):

$$\boldsymbol{x} \colon U = (0, \infty) \times (0, 2\pi) \longrightarrow P, \quad \boldsymbol{x}(r, \vartheta) = (r \cos \vartheta, r \sin \vartheta, 0).$$

Show that the map $f \colon P \setminus \{(0, 0, 0)\} \longrightarrow S$ defined on $\boldsymbol{x}(U)$ by

$$f(\boldsymbol{x}(r, \vartheta)) = \frac{1}{n}\big(r \cos n\vartheta, r \sin n\vartheta, br\big)$$

is a local isometry on $\boldsymbol{x}(U)$.

Solution:

We have

$$\boldsymbol{x}_r = (\cos \vartheta, \sin \vartheta, 0) \quad \text{and} \quad \boldsymbol{x}_\vartheta = (-r \sin \vartheta, r \cos \vartheta, 0),$$

so that the coefficients of the first fundamental form of P with respect to the parametrization $\boldsymbol{x}$ *(polar coordinates — parametrized $P \setminus \{\mathbf{0}\}$)* are

$$E(r, \vartheta) = 1, \qquad F(r, \vartheta) = 0 \quad \text{and} \quad G(r, \vartheta) = r^2.$$

Now calculate

$$f_r := (f \circ \boldsymbol{x})_r = \frac{1}{n}(\cos(n\vartheta), \sin(n\vartheta), b) \quad \text{and} \quad f_\vartheta := (f \circ \boldsymbol{x})_\vartheta = (-r\sin(n\vartheta), r\cos(n\vartheta), 0),$$

so that

$$\widetilde{E} := \langle f_r, f_r \rangle = \frac{1+b^2}{n^2}, \qquad \widetilde{F} := \langle f_r, f_\vartheta \rangle = 0 \quad \text{and} \quad \widetilde{G} := \langle f_\vartheta, f_\vartheta \rangle = r^2.$$

By assumption, $(1+b^2)/n^2 = 1$, so that $\widetilde{E} = E$, $\widetilde{F} = F$ and $\widetilde{G} = G$, hence f is a local isometry by Proposition 8.15.

11.3. Let S_1, S_2, S_3 be regular surfaces.

(a) Suppose that $f\colon S_1 \longrightarrow S_2$ and $g\colon S_2 \longrightarrow S_3$ are local isometries. Prove that $g \circ f\colon S_1 \longrightarrow S_3$ is a local isometry.

(b) Suppose that $f\colon S_1 \longrightarrow S_2$ and $g\colon S_2 \longrightarrow S_3$ are conformal maps with conformal factors $\lambda\colon S_1 \longrightarrow (0,\infty)$ and $\mu\colon S_2 \longrightarrow (0,\infty)$, respectively. Prove that $g \circ f\colon S_1 \longrightarrow S_3$ is a conformal map and calculate its conformal factor. (The conformal factor of a conformal map is the function appearing as factor in front of the inner product in the definition.)

(c) Let f and g be conformal maps with conformal factors λ and μ as in the previous part. Find a condition on λ and μ such that $g \circ f$ is a *(local) isometry.*

Solution:

(a) By the definition of a local isometry,

$$\langle d_{p_1} f(\boldsymbol{v}_1), d_{p_1} f(\boldsymbol{w}_1) \rangle_{f(p_1)} = \langle \boldsymbol{v}_1, \boldsymbol{w}_1 \rangle_{p_1} \quad \text{and} \quad \langle d_{p_2} g(\boldsymbol{v}_2), d_{p_2} f(\boldsymbol{w}_2) \rangle_{g(p_2)} = \langle \boldsymbol{v}_2, \boldsymbol{w}_2 \rangle_{p_2}$$

for all $p_1 \in S_1$, $\boldsymbol{v}_1, \boldsymbol{w}_1 \in T_{p_1} S_1$ and $p_2 \in S_2$, $\boldsymbol{v}_2, \boldsymbol{w}_2 \in T_{p_2} S_2$.
This notation is also already part ot the solution: applying these two equations with $p_2 = f(p_1)$, $\boldsymbol{v}_2 = d_{p_1} f(\boldsymbol{v}_1)$ and $\boldsymbol{w}_2 = d_{p_1} f(\boldsymbol{w}_1)$, and using the chain rule

$$d_{p_1}(g \circ f)(\boldsymbol{w}_1) = d_{f(p_1)} g(d_{p_1} f(\boldsymbol{w}_1))$$

for all $p_1 \in S_1$ and $\boldsymbol{w}_1 \in T_{p_1} S_1$, we obtain

$$\begin{aligned} \langle d_{p_1}(g \circ f)(\boldsymbol{v}_1), d_{p_1}(g \circ f) f(\boldsymbol{w}_1) \rangle_{(g \circ f)(p_1)} &= \langle d_{f(p_1)} g(d_{p_1} f(\boldsymbol{v}_1)), d_{f(p_1)} g(d_{p_1} f(\boldsymbol{w}_1)) \rangle_{(g(f(p_1))} \\ &= \langle d_{p_1} f(\boldsymbol{v}_1), d_{p_1} f(\boldsymbol{w}_1) \rangle_{f(p_1)} \\ &= \langle \boldsymbol{v}_1, \boldsymbol{w}_1 \rangle_{p_1} \end{aligned}$$

using the chain rule for the first, the isometry of g for the second and the isometry of f for the last equality. Hence we have shown that $g \circ f)$ is a local isometry using the definition.

(b) The proof is almost the same as the one of the first part: since f and g are conformal maps, we have

$$\langle d_{p_1} f(\boldsymbol{v}_1), d_{p_1} f(\boldsymbol{w}_1) \rangle_{f(p_1)} = \lambda(p_1) \langle \boldsymbol{v}_1, \boldsymbol{w}_1 \rangle_{p_1} \quad \text{and} \quad \langle d_{p_2} g(\boldsymbol{v}_2), d_{p_2} f(\boldsymbol{w}_2) \rangle_{g(p_2)} = \mu(p_2) \langle \boldsymbol{v}_2, \boldsymbol{w}_2 \rangle_{p_2}$$

for all $p_1 \in S_1$, $\boldsymbol{v}_1, \boldsymbol{w}_1 \in T_{p_1} S_1$ and $p_2 \in S_2$, $\boldsymbol{v}_2, \boldsymbol{w}_2 \in T_{p_2} S_2$.
Applying these two equations with $p_2 = f(p_1)$, $\boldsymbol{v}_2 = d_{p_1} f(\boldsymbol{v}_1)$ and $\boldsymbol{w}_2 = d_{p_1} f(\boldsymbol{w}_1)$, and using again the chain rule we obtain

$$\begin{aligned} \langle d_{p_1}(g \circ f)(\boldsymbol{v}_1), d_{p_1}(g \circ f) f(\boldsymbol{w}_1) \rangle_{(g \circ f)(p_1)} &= \langle d_{f(p_1)} g(d_{p_1} f(\boldsymbol{v}_1)), d_{f(p_1)} g(d_{p_1} f(\boldsymbol{w}_1)) \rangle_{(g(f(p_1))} \\ &= \mu(f(p_1)) \langle d_{p_1} f(\boldsymbol{v}_1), d_{p_1} f(\boldsymbol{w}_1) \rangle_{f(p_1)} \\ &= \mu(f(p_1)) \lambda(p_1) \langle \boldsymbol{v}_1, \boldsymbol{w}_1 \rangle_{p_1} \end{aligned}$$

using the chain rule for the first, the conformality of g for the second and the conformality of f for the last equality. Hence we have shown that $g \circ f)$ is a conformal map with conformal factor

$$(\mu \circ f) \cdot \lambda \colon S_1 \longrightarrow (0, \infty, \qquad p_1 \mapsto \mu(f(p_1))\lambda(p_1).$$

(c) The third part is again rather trivial. We want that $(\mu \circ f) \cdot \lambda$ equals the constant function 1 on S_1, i.e., that

$$\mu(f(p_1)) = \frac{1}{\lambda(p_1)}$$

for all $p_1 \in S_1$. In particular, we do not need any restriction on the behaviour of μ outside the range $f(S_1)$ of f.

11.4. Let S be the surface of revolution parametrized by

$$\boldsymbol{x}(u, v) = \Big(\cos v \cos u, \cos v \sin u, -\sin v + \log \tan\Big(\frac{\pi}{4} + \frac{v}{2}\Big)\Big),$$

where $0 < u < 2\pi, 0 < v < \pi/2$. Let S_1 be the region

$$S_1 = \{\, \boldsymbol{x}(u, v) \,|\, 0 < u < \pi, 0 < v < \pi/2 \,\}$$

and let S_2 be the region

$$S_2 = \{\, \boldsymbol{x}(u, v) \,|\, 0 < u < 2\pi, \pi/3 < v < \pi/2 \,\}.$$

Show that the map

$$\boldsymbol{x}(u, v) \mapsto \boldsymbol{x}\Big(2u, \arccos\Big(\frac{1}{2}\cos v\Big)\Big)$$

is an isometry from S_1 onto S_2.

Solution:

The map $f \colon S_1 \longrightarrow S_2$ is actually a bijection (see below), so one can prove that it gives rise to a local parametrization; we will use Prop. 8.15 from the lectures and show that the coefficients E, F and G (w.r.t. the parametrization $\boldsymbol{x}$) are the same as the coefficients $\widetilde{E}$, $\widetilde{F}$ and $\widetilde{G}$ w.r.t the parametrization

$$\widetilde{\boldsymbol{x}}(u, v) := \boldsymbol{x}\Big(2u, \arccos\Big(\frac{1}{2}\cos v\Big)\Big).$$

Let us calculate E, F and G first. We have

$$\boldsymbol{x}_u = (-\cos v \sin u, \cos v \cos u, 0), \qquad \boldsymbol{x}_v = (-\sin v \cos u, -\sin v \sin u, -\cos v + 1/\cos v),$$

as the derivative of g with $g(v) = -\sin v + \log \tan(\pi/4 + v/2)$ is

$$\begin{aligned} g'(v) &= -\cos v + \frac{1}{2}\Big(\tan\Big(\frac{\pi}{4} + \frac{v}{2}\Big)\Big)^{-1} \tan'\Big(\frac{\pi}{4} + \frac{v}{2}\Big) \\ &= -\cos v + \frac{\cos(\pi/4 + v/2)}{2\sin(\pi/4 + v/2)\cos^2(\pi/4 + v/2)} \\ &= -\cos v + \frac{1}{\sin(\pi/2 + v)} \\ &= -\cos v + \frac{1}{\cos v} = \frac{-\cos^2 v + 1}{\cos v} = \frac{\sin^2 v}{\cos v}. \end{aligned}$$

In particular,

$$
\begin{aligned}
E(u,v) &= \cos^2 v, \qquad\qquad F(u,v) = 0,\\
G(u,v) &= \sin^2 v + \Big(\frac{1}{\cos v} - \cos v\Big)^2\\
&= 1 - \cos^2 v + \frac{1}{\cos^2 v} - 2 + \cos^2 v\\
&= \frac{1}{\cos^2 v} - 1 = \frac{1-\cos^2 v}{\cos^2 v} = \tan^2 v
\end{aligned}
$$

Let us now calculate the coefficients w.r.t. the parametrization $\widetilde{\boldsymbol{x}}$ (make sure you use the arguments of the functions correctly):

$$
\begin{aligned}
f_u(u,v) = \widetilde{\boldsymbol{x}}_u(u,v) &= 2\boldsymbol{x}_u(2u, \arccos((\cos v)/2))\\
f_v(u,v) = \widetilde{\boldsymbol{x}}_v(u,v) &= \varphi'(v)\boldsymbol{x}_v(2u, \arccos((\cos v)/2))\\
&= \frac{\sin v}{2\sqrt{1-(\cos^2 v)/4}}\boldsymbol{x}_v(2u, \arccos((\cos v)/2))
\end{aligned}
$$

since the derivative of φ given by $\varphi(v) = \arccos((\cos v)/2)$ is

$$
\varphi'(v) = \frac{1}{2}(-\sin v)\arccos'((\cos v)/2) = \frac{\sin v}{2\sqrt{1-(\cos^2 v)/4}}.
$$

In particular,

$$
\begin{aligned}
\langle f_u(u,v), f_u(u,v)\rangle &= \widetilde{E}(u,v) = 4E(2u, \arccos((\cos v)/2)),\\
\langle f_u(u,v), f_v(u,v)\rangle &= \widetilde{F}(u,v) = 2\varphi'(v)F(\dots) = 0\\
\langle f_v(u,v), f_v(u,v)\rangle &= \widetilde{G}(u,v) = \frac{\sin^2 v}{4(1-(\cos^2 v)/4)}G(2u, \arccos((\cos v)/2)).
\end{aligned}
$$

Let us now simplify these expressions in order to obtain $E = \widetilde{E}$ and $G = \widetilde{G}$ ($F = \widetilde{F} = 0$ is aready clear):

$$
\begin{aligned}
\widetilde{E}(u,v) &= 4E(2u, \arccos((\cos v)/2))\\
&= 4\cos^2 \arccos((\cos v)/2))\\
&= 4\big((\cos v)/2\big)^2 = \cos^2 v = E(u,v),
\end{aligned}
$$

as $\cos(\arccos z) = z$ for $z \in [-1,1]$.

Moreover,

$$
\begin{aligned}
\widetilde{G}(u,v) &= \frac{\sin^2 v}{4(1-(\cos^2 v)/4)}G(2u, \arccos((\cos v)/2))\\
&= \frac{\sin^2 v}{4(1-(\cos^2 v)/4)}\Big(\frac{1}{\cos^2(\arccos((\cos v)/2))} - 1\Big)\\
&= \frac{\sin^2 v}{4(1-(\cos^2 v)/4)}\Big(\frac{1}{(\cos^2 v)/4} - 1\Big)\\
&= \frac{\sin^2 v}{4(1-(\cos^2 v)/4)}\Big(\frac{1-\cos^2 v/4}{(\cos^2 v)/4}\Big)\\
&= \frac{\sin^2 v}{\cos^2 v} = G(u,v)
\end{aligned}
$$

(where we use the expression of $G(u,v)$ involving $\cos v$ only for the second equality).

Hence, by Proposition 8.15, f is a local isometry.

For f being an isometry, we also need that $f\colon S_1 \longrightarrow S_2$ is a bijection: Basically, we map $(u,v) \in U_1 = (0,\pi)\times(0,\pi/2)$ onto $\Phi(u,v) := (2u, \arccos((\cos v)/2)) \in U_2 = (0,2\pi)\times(\pi/3,\pi/2)$, and as

$$\psi\colon (0,\pi) \longrightarrow (0,2\pi), \quad \psi(u) = 2u$$

and

$$\varphi\colon (0,\pi/2) \longrightarrow (\pi/3,\pi/2), \quad \varphi(v) = \arccos((\cos v)/2)$$

are both bijections, $\Phi\colon U_1 \longrightarrow U_2$ is a bijection and hence also $f = \boldsymbol{x}\circ\Phi\circ\boldsymbol{x}^{-1}$.

12.1. $(*)$ Let S be a surface of revolution. Prove that any rotation about the axis of revolution is an isometry of S.

Solution:

Let S be parametrised by $\boldsymbol{x}\colon U \longrightarrow S$ with

$$\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

and $U = (-\pi,\pi)\times J$ or $U = (0,2\pi)\times J$, where $f\colon J \longrightarrow (0,\infty)$ and $g\colon J \longrightarrow \mathbb{R}$ are the functions of the generating curve given by $v \mapsto (f(v),0,g(v))$. We know that

$$E(u,v) = f(v)^2, \qquad F(u,v) = 0, \qquad G(u,v) = f'(v)^2 + g'(v)^2$$

The rotation R by an angle ϑ around the symmetry axis is define by

$$R(\boldsymbol{x}(u,v)) = \boldsymbol{x}(u+\vartheta, v)$$

(for appropriate parameter values $(u,v)\in U$ such that $(u+\vartheta,v)\in U$). Then we have

$$\begin{aligned}
R_u(u,v) &= (R\circ\boldsymbol{x})_u(u,v) = \boldsymbol{x}_u(u+\vartheta,v)\\
R_v(u,v) &= (R\circ\boldsymbol{x})_v(u,v) = \boldsymbol{x}_v(u+\vartheta,v),
\end{aligned}$$

hence

$$\begin{aligned}
\widetilde{E}(u,v) = \langle R_u(u,v), R_u(u,v)\rangle &= \boldsymbol{x}_u(u+\vartheta,v)\cdot\boldsymbol{x}_u(u+\vartheta,v) = E(u+\vartheta,v) = f(v)^2\\
&= E(u,v)\\
\widetilde{F}(u,v) = \langle R_u(u,v), R_v(u,v)\rangle &= \boldsymbol{x}_u(u+\vartheta,v)\cdot\boldsymbol{x}_v(u+\vartheta,v) = 0 = F(u,v)\\
\widetilde{G}(u,v) = \langle R_v(u,v), R_v(u,v)\rangle &= \boldsymbol{x}_v(u+\vartheta,v)\cdot\boldsymbol{x}_v(u+\vartheta,v)\\
&= G(u+\vartheta,v) = f'(v)^2 + g'(v)^2 = G(u,v)
\end{aligned}$$

(in other words, the coefficients do not depend on the angle variable u).

Hence, f is a local isometry. Moreover, $R = R_\vartheta\colon S \longrightarrow S$ is obviously a bijection, so it is a global isometry.

Alternatively, one can note that $R = R_\vartheta\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear orthogonal map, so its differential $d_pR_\vartheta = R_\vartheta$ preserves lengths of all tangent (to $\mathbb{R}^3$) vectors. This means that R_ϑ is a global isometry of any surface onto its image. Now, since $R_\vartheta(S) = S$, R_ϑ is a global isometry of S.

12.2. The disc model of the hyperbolic plane.

Let $\mathbb{D}$ denote the unit disc $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ with first fundamental form

$$\widetilde{E} = \widetilde{G} = \frac{4}{(1 - x^2 - y^2)^2}, \quad \widetilde{F} = 0.$$

Let $\mathbb{H}$ be the hyperbolic plane with coordinates $(u, v) \in \mathbb{R} \times (0, \infty)$ and first fundamental form

$$E = G = \frac{1}{v^2}, \quad F = 0.$$

Show that the map $\boldsymbol{f} : \mathbb{H} \longrightarrow \mathbb{D}$ given by

$$\boldsymbol{f}(z) = \frac{z - \mathrm{i}}{z + \mathrm{i}}, \qquad z = u + \mathrm{i}v \in \mathbb{H},$$

is an isometry.

Solution: We can consider $(x, y) = (\mathrm{Re}(\boldsymbol{f}), \mathrm{Im}(\boldsymbol{f}))$ as a coordinate system on $\mathbb{D}$ (the bijectivity can be checked easily, please also check that the differential is non-degenerate everywhere).

If we take a tangent vector $\boldsymbol{w} = (a, b) \in T_{(u,v)}\mathbb{H}$, then the square of its length is equal to

$$\langle \boldsymbol{w}, \boldsymbol{w} \rangle_{(u,v)} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a^2 E + 2abF + b^2 G = \frac{a^2 + b^2}{v^2} = \frac{\langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\mathrm{Eucl}}}{v^2}$$

by the definition of the coefficients of the first fundamental form, where $\langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\mathrm{Eucl}}$ is the Euclidean dot product.

The differential of $\boldsymbol{f}$ can be written as

$$d_{(u,v)}\boldsymbol{f} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} & \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial u} & \frac{\partial y(u,v)}{\partial v} \end{pmatrix}, = \begin{pmatrix} \boldsymbol{f}_u & \boldsymbol{f}_v \end{pmatrix},$$

where

$$\boldsymbol{f}_u = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \end{pmatrix} = d_{(u,v)}\boldsymbol{f}((1,0)), \qquad \boldsymbol{f}_v = \begin{pmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \end{pmatrix} = d_{(u,v)}\boldsymbol{f}((0,1)).$$

Then

$$d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}) = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} & \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial u} & \frac{\partial y(u,v)}{\partial v} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a\boldsymbol{f}_u + b\boldsymbol{f}_v.$$

The square of the length of $d_{(u,v)}\boldsymbol{f}(\boldsymbol{w})$ is then can be computed as

$$\langle d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}), d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}) \rangle_{\boldsymbol{f}(u,v)} = (d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}))^T \begin{pmatrix} \widetilde{E}(u,v) & \widetilde{F}(u,v) \\ \widetilde{F}(u,v) & \widetilde{G}(u,v) \end{pmatrix} (d_{(u,v)}\boldsymbol{f}(\boldsymbol{w})) =$$

$$\frac{4\langle d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}), d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}) \rangle_{\mathrm{Eucl}}}{(1 - x^2 - y^2)^2} = \frac{4}{(1 - x^2 - y^2)^2}(a^2 \langle \boldsymbol{f}_u, \boldsymbol{f}_u \rangle_{\mathrm{Eucl}} + 2ab\langle \boldsymbol{f}_u, \boldsymbol{f}_v \rangle_{\mathrm{Eucl}} + b^2 \langle \boldsymbol{f}_v, \boldsymbol{f}_v \rangle_{\mathrm{Eucl}}).$$

To show that $\boldsymbol{f}$ is an isometry, We need to show that $\langle \boldsymbol{w}, \boldsymbol{w} \rangle_{(u,v)} = \langle d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}), d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}) \rangle_{\boldsymbol{f}(u,v)}$.

Writing

$$\begin{aligned} x+iy &= \boldsymbol{f}(u+iv) \\ &= \frac{u+iv-i}{u+iv+i} \\ &= \frac{(u+iv-i)(u-iv-i)}{u^2+(v+1)^2} \\ &= \frac{u^2+v^2-1}{u^2+(v+1)^2}+i\frac{-2u}{u^2+(v+1)^2}, \end{aligned}$$

we have

$$\boldsymbol{f}(u,v)=(x(u,v),y(u,v))=\frac{1}{u^2+(v+1)^2}(u^2+v^2-1,-2u).$$

In particular, we can calculate that

$$1-x^2-y^2=1-\frac{(u^2+v^2-1)^2+(-2u)^2}{(u^2+(v+1)^2)^2}=\frac{4v}{u^2+(v+1)^2}.$$

Taking partial derivatives gives

$$\begin{aligned} \boldsymbol{f}_u &= \frac{1}{(u^2+(v+1)^2)^2}(4u(v+1),2u^2-2(v+1)^2), \\ \boldsymbol{f}_v &= \frac{1}{(u^2+(v+1)^2)^2}(-2u^2+2(v+1)^2,4u(v+1)). \end{aligned}$$

Computing the (Euclidean) inner products of the vectors above, we obtain

$$\begin{aligned} \boldsymbol{f}_u\cdot\boldsymbol{f}_u &= \frac{4}{(u^2+(v+1)^2)^4}(4u^2(v+1)^2+(u^2-(v+1)^2)^2)=\frac{4}{(u^2+(v+1)^2)^2}, \\ \boldsymbol{f}_u\cdot\boldsymbol{f}_v &= 0, \\ \boldsymbol{f}_v\cdot\boldsymbol{f}_v &= \frac{4}{(u^2+(v+1)^2)^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} 4\frac{\boldsymbol{f}_u\cdot\boldsymbol{f}_u}{(1-x^2-y^2)^2}=4\frac{\frac{4}{(u^2+(v+1)^2)^2}}{\left(\frac{4v}{u^2+(v+1)^2}\right)^2}=\frac{1}{v^2} &= E, \\ 4\frac{\boldsymbol{f}_u\cdot\boldsymbol{f}_v}{(1-x^2-y^2)^2}=0 &= F, \\ 4\frac{\boldsymbol{f}_v\cdot\boldsymbol{f}_v}{(1-x^2-y^2)^2}=\frac{1}{v^2} &= G, \end{aligned}$$

and thus

$$\langle d_{(u,v)}\boldsymbol{f}(\boldsymbol{w}),d_{(u,v)}\boldsymbol{f}(\boldsymbol{w})\rangle_{\boldsymbol{f}(u,v)}=a^2E+2abF+b^2G=\langle\boldsymbol{w},\boldsymbol{w}\rangle_{(u,v)}$$

(compare to Proposition 8.15 from the lectures).

12.3. Hyperboloid model of the hyperbolic plane.

Let $Q:\mathbb{R}^3\to\mathbb{R}$ be the quadratic form defined by

$$Q(x_1,x_2,x_3)=x_1^2+x_2^2-x_3^2, \qquad (x_1,x_2,x_3)\in\mathbb{R}^3$$

(the quadratic space $(\mathbb{R}^3, Q)$ is usually denoted by $\mathbb{R}^{2,1}$). Let

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, Q(x_1, x_2, x_3) = -1\}$$

(i.e. S is a hyperboloid of two sheets).

Recall that the *induced quadratic form* $I_{\boldsymbol{p}}$ on each tangent plane $T_{\boldsymbol{p}}S$ is defined by $I_{\boldsymbol{p}}(\boldsymbol{w}) = Q(\boldsymbol{w})$ for every $\boldsymbol{w} \in T_{\boldsymbol{p}}(S)$. Show that $I_{\boldsymbol{p}}$ is positive definite and that the map $f : \mathbb{D} \to S$ from the disc model of the hyperbolic plane (see the previous exercise) defined by

$$\boldsymbol{f}(x, y) = \frac{1}{1 - x^2 - y^2}\,(2x, 2y, 1 + x^2 + y^2), \qquad (x, y) \in \mathbb{D},$$

maps $\mathbb{D}$ isometrically onto the component of S for which $x_3 > 0$.

Solution:

Note that $\boldsymbol{f}$ is parametrization of the "upper" part of S (please check bijectivity!). In particular,

$$\begin{aligned}
\boldsymbol{f}_x &= \frac{2}{(1 - x^2 - y^2)^2}\,((1 + x^2 - y^2), 2xy, 2x), \\
\boldsymbol{f}_y &= \frac{2}{(1 - x^2 - y^2)^2}\,(2xy, (1 - x^2 + y^2), 2y),
\end{aligned}$$

which implies that

$$\begin{aligned}
\widetilde{E} &= Q(\boldsymbol{f}_x) = \frac{4}{(1 - x^2 - y^2)^4}\,((1 + x^2 - y^2)^2 + (2xy)^2 - (2x)^2) = \frac{4}{(1 - x^2 - y^2)^2} = E, \\
\widetilde{F} &= 0, \\
\widetilde{G} &= Q(\boldsymbol{f}_y) = \frac{4}{(1 - x^2 - y^2)^2} = G.
\end{aligned}$$