Durham University
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Differential Geometry 16.01.2020

Homework 11-12
Starred problems due on Thursday, 30th January.

Isometries and conformal maps - 1

11.1. Let $a > 0$. Construct explicitly a local isometry from the plane $P = \{ (u, v, 0) \in \mathbb{R}^3 \mid u, v \in \mathbb{R} \}$ onto the cylinder $S = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2 \}$.

11.2. $(*)$ Let b be a positive number such that $\sqrt{1 + b^2}$ is an integer n. Let S be the circular cone obtained by rotating the curve given by $\boldsymbol{\alpha}(v) = (v, 0, bv)$, $v > 0$, about the z-axis. Let the coordinate xy-plane P be parametrized by polar coordinates (r, ϑ):

$$\boldsymbol{x}\colon U = (0, \infty) \times (0, 2\pi) \longrightarrow P, \quad \boldsymbol{x}(r, \vartheta) = (r \cos \vartheta, r \sin \vartheta, 0).$$

Show that the map $f\colon P \setminus \{(0, 0, 0)\} \longrightarrow S$ defined on $\boldsymbol{x}(U)$ by

$$f(\boldsymbol{x}(r, \vartheta)) = \frac{1}{n}\big(r \cos n\vartheta, r \sin n\vartheta, br\big)$$

is a local isometry on $\boldsymbol{x}(U)$.

11.3. Let S_1, S_2, S_3 be regular surfaces.

(a) Suppose that $f\colon S_1 \longrightarrow S_2$ and $g\colon S_2 \longrightarrow S_3$ are local isometries. Prove that $g \circ f\colon S_1 \longrightarrow S_3$ is a local isometry.

(b) Suppose that $f\colon S_1 \longrightarrow S_2$ and $g\colon S_2 \longrightarrow S_3$ are conformal maps with conformal factors $\lambda\colon S_1 \longrightarrow (0, \infty)$ and $\mu\colon S_2 \longrightarrow (0, \infty)$, respectively. Prove that $g \circ f\colon S_1 \longrightarrow S_3$ is a conformal map and calculate its conformal factor. (The conformal factor of a conformal map is the function appearing as factor in front of the inner product in the definition.)

(c) Let f and g be conformal maps with conformal factors λ and μ as in the previous part. Find a condition on λ and μ such that $g \circ f$ is a *(local) isometry.*

11.4. Let S be the surface of revolution parametrized by

$$\boldsymbol{x}(u, v) = \Big(\cos v \cos u, \cos v \sin u, -\sin v + \log \tan\Big(\frac{\pi}{4} + \frac{v}{2}\Big)\Big),$$

where $0 < u < 2\pi, 0 < v < \pi/2$. Let S_1 be the region

$$S_1 = \{ \vec{x}(u, v) \mid 0 < u < \pi, 0 < v < \pi/2 \}$$

and let S_2 be the region

$$S_2 = \{ \vec{x}(u, v) \mid 0 < u < 2\pi, \pi/3 < v < \pi/2 \}.$$

Show that the map

$$\boldsymbol{x}(u, v) \mapsto \boldsymbol{x}\Big(2u, \arccos\Big(\frac{1}{2} \cos v\Big)\Big)$$

is an isometry from S_1 onto S_2.

Isometries and conformal maps - 2

12.1. (*) Let S be a surface of revolution. Prove that any rotation about the axis of revolution is an isometry of S.

12.2. The disc model of the hyperbolic plane.
Let $\mathbb{D}$ denote the unit disc $\{(x,y)\in\mathbb{R}^2 \,|\, x^2+y^2<1\}$ with first fundamental form

$$\widetilde{E}=\widetilde{G}=\frac{4}{(1-x^2-y^2)^2}, \qquad \widetilde{F}=0.$$

Let $\mathbb{H}$ be the hyperbolic plane with coordinates $(u,v)\in\mathbb{R}\times(0,\infty)$ and first fundamental form

$$E=G=\frac{1}{v^2}, \quad F=0.$$

Show that the map $f\colon \mathbb{H}\longrightarrow\mathbb{D}$ given by

$$f(z)=\frac{z-\mathrm{i}}{z+\mathrm{i}}, \qquad z=u+\mathrm{i}v\in\mathbb{H},$$

is an isometry.

12.3. Hyperboloid model of the hyperbolic plane.
Let $Q:\mathbb{R}^3\to\mathbb{R}$ be the quadratic form defined by

$$Q(x_1,x_2,x_3)=x_1^2+x_2^2-x_3^2, \qquad\qquad (x_1,x_2,x_3)\in\mathbb{R}^3$$

(the quadratic space $(\mathbb{R}^3,Q)$ is usually denoted by $\mathbb{R}^{2,1}$). Let

$$S=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \,|\, Q(x_1,x_2,x_3)=-1\}$$

(i.e. S is a hyperboloid of two sheets).

Recall that the *induced quadratic form* $I_{\boldsymbol{p}}$ on each tangent plane $T_{\boldsymbol{p}}S$ is defined by $I_{\boldsymbol{p}}(\boldsymbol{w})=Q(\boldsymbol{w})$ for every $\boldsymbol{w}\in T_{\boldsymbol{p}}(S)$. Show that $I_{\boldsymbol{p}}$ is positive definite and that the map $f:\mathbb{D}\to S$ from the disc model of the hyperbolic plane (see the previous exercise) defined by

$$f(x,y)=\frac{1}{1-x^2-y^2}\,(2x,2y,1+x^2+y^2), \qquad\qquad (x,y)\in\mathbb{D},$$

maps $\mathbb{D}$ isometrically onto the component of S for which $x_3>0$.