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Solutions 13-14

13.1. A local parametrization $\boldsymbol{x}$ of a surface S in $\mathbb{R}^3$ is called *orthogonal* provided $F = 0$ (so $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are orthogonal at each point). It is called *principal* if $F = 0$ and $M = 0$, where E, F, G (resp. L, M, N) are the coefficients of the first (resp. second) fundamental form.

(a) Let $\boldsymbol{x}$ be an *orthogonal* parametrization. Show that, at any point $p = \boldsymbol{x}(u, v)$ on S,

$$-d\boldsymbol{N}_p(\boldsymbol{x}_u) = \frac{L}{E}\boldsymbol{x}_u + \frac{M}{G}\boldsymbol{x}_v, \qquad -d\boldsymbol{N}_p(\boldsymbol{x}_v) = \frac{M}{E}\boldsymbol{x}_u + \frac{N}{G}\boldsymbol{x}_v,$$

where $\boldsymbol{N}$ denotes the Gauss map.

(b) Assume now that the parametrization is *principal*. Show that $\kappa_1 = L/E$ and $\kappa_2 = N/G$ are the principal curvatures. Calculate the Gauss and mean curvature in terms of E, G, L, N. Determine the principal directions.

Solution:

(a) Since $d_p\boldsymbol{N}$ maps T_pS into T_pS, we can express $-d_p\boldsymbol{N}(\boldsymbol{x}_u)$ and $-d_p\boldsymbol{N}(\boldsymbol{x}_v)$ as a linear combination of $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, i.e.,

$$-d_p\boldsymbol{N}(\boldsymbol{x}_u) = a\boldsymbol{x}_u + b\boldsymbol{x}_v \quad \text{and} \quad -d_p\boldsymbol{N}(\boldsymbol{x}_v) = c\boldsymbol{x}_u + d\boldsymbol{x}_v.$$

Multiplying both equations with $\cdot\boldsymbol{x}_u$ and $\cdot\boldsymbol{x}_v$ gives (using the definitions of the coefficients of the first and second fundamental forms and the equalities $\boldsymbol{N}_u \cdot \boldsymbol{x}_u + \boldsymbol{N} \cdot \boldsymbol{x}_{uu} = 0$ etc.)

$$L = aE + bF, \quad M = aF + bG, \quad M = cE + dF, \quad N = cF + dG,$$

and, since $F = 0$,

$$a = \frac{L}{E}, \quad b = \frac{M}{G}, \quad c = \frac{M}{E}, \quad d = \frac{N}{G},$$

i.e., the desired equation.

(b) If $M = 0$, then the equations from the first part are

$$-d\boldsymbol{N}_p(\boldsymbol{x}_u) = \frac{L}{E}\boldsymbol{x}_u \quad \text{and} \quad -d\boldsymbol{N}_p(\boldsymbol{x}_v) = \frac{N}{G}\boldsymbol{x}_v.$$

Therefore, $\boldsymbol{x}_u$ is an eigenvector with eigenvalue L/E, as well as $\boldsymbol{x}_v$ with eigenvalue N/G. Hence the principal, Gauss and mean curvatures are

$$\kappa_1 = \frac{L}{E}, \quad \kappa_2 = \frac{N}{G}, \quad K = \kappa_1\kappa_2 = \frac{LN}{EG}, \quad H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{L}{2E} + \frac{N}{2G} = \frac{LG + NE}{2EG}.$$

13.2. Calculation of the Weingarten map directly for surfaces of revolution

Let $f\colon J \longrightarrow (0, \infty)$ and $g\colon J \longrightarrow \mathbb{R}$ be smooth functions on some open interval J in $\mathbb{R}$ and let $\boldsymbol{\alpha}\colon J \longrightarrow \mathbb{R}^3$ be a space curve given by $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$. Assume that this curve is parametrized by arc length. Let S be the surface of revolution obtained by rotating $\boldsymbol{\alpha}$ around the z-axis.

(a) Find suitable parametrizations $\boldsymbol{x}\colon U_i \longrightarrow S$ of S and determine parameter domains U_1 and U_2 covering the whole surface S. Calculate the normal vector $\boldsymbol{N}$ at $\boldsymbol{x}(u,v)$

(b) Express $a,b,c,d \in \mathbb{R}$ in $-d\boldsymbol{N}_p(\boldsymbol{x}_u) = a\boldsymbol{x}_u + b\boldsymbol{x}_v$ and $-d\boldsymbol{N}_p(\boldsymbol{x}_v) = c\boldsymbol{x}_u + d\boldsymbol{x}_v$ in terms of f and g.

(c) Calculate the principal directions and principal curvatures.

(d) Calculate the Gauss and mean curvatures.

(e) Compare your results with Example 9.13 from the lectures.

Solution: The generating curve is parametrized by arc length, so $(f')^2 + (g')^{=}1$.

(a) The standard parametrization of a surface of revolution is given by

$$\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)), \qquad (u,v) \in U$$

where $U = U_1$ or $U = U_2$ and (for example)

$$U_1 = (-\pi, \pi) \times J, \qquad U_2 = (0, 2\pi) \times J,$$

so that the first (angular) variable u covers all angles.
Make sure you understand why we need (at least) two parameter sets U_1 and U_2.
Moreover, (f, g have the argument v, and cos, sin have the argument u)

$$\boldsymbol{x}_u = (-f\sin, f\cos, 0), \qquad \boldsymbol{x}_v = (f'\cos, f'\sin, g'),$$

hence $\boldsymbol{x}_u \times \boldsymbol{x}_v = (g'\cos, g'\sin, -f')$. Since the generating curve is parametrized by arc length, $\boldsymbol{x}_u \times \boldsymbol{x}_v$ is a unit vector, so

$$\boldsymbol{N} = (g'\cos, g'\sin, -f').$$

Moreover,

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u = f^2, \qquad F = 0, \qquad G = (f')^2 + (g')^2 = 1.$$

We also need (later on) the coefficients of the second fundamental form, so we calculate

$$\boldsymbol{x}_{uu} = (-f\cos, -f\sin, 0), \qquad \boldsymbol{x}_{uv} = (-f'\sin, f'\cos, 0), \qquad \boldsymbol{x}_{vv} = (f''\cos, f''\sin, g'')$$

so that

$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = -fg', \qquad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = 0, \qquad N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = f''g' - f'g''$$

(b) We multiply both equations with $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, so that

$$\begin{aligned} L &= -d_p\boldsymbol{N}(\boldsymbol{x}_u) \cdot \boldsymbol{x}_u = aE + bF, & M &= -d_p\boldsymbol{N}(\boldsymbol{x}_u) \cdot \boldsymbol{x}_v = aF + bG, \\ M &= -d_p\boldsymbol{N}(\boldsymbol{x}_v) \cdot \boldsymbol{x}_u = cE + dF, & N &= -d_p\boldsymbol{N}(\boldsymbol{x}_v) \cdot \boldsymbol{x}_v = cF + dG, \end{aligned}$$

where we used the equalities $\boldsymbol{N}_u \cdot \boldsymbol{x}_u + \boldsymbol{N} \cdot \boldsymbol{x}_{uu} = 0$ etc.
The above equations simplify to

$$\begin{aligned} L &= aE, & M &= bG, \\ M &= cE, & N &= dG. \end{aligned}$$

If $F = 0$, then

$$a = \frac{L}{E}, \qquad b = \frac{M}{G}, \qquad c = \frac{M}{E}, \qquad d = \frac{N}{G}.$$

If, in addition, $M = 0$, then

$$a = \frac{L}{E}, \qquad b = 0, \qquad c = 0, \qquad d = \frac{N}{G}.$$

(c) We have (using the above expressions for a, b, c and d)

$$-d_p\boldsymbol{N}(\boldsymbol{x}_u) = \frac{L}{E}\boldsymbol{x}_u \qquad \text{and} \qquad -d_p\boldsymbol{N}(\boldsymbol{x}_v) = \frac{N}{G}\boldsymbol{x}_v,$$

hence the basis vectors $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are eigenvectors (principal directions) with eigenvalues (principal curvatures)

$$\kappa_1 = \frac{L}{E} = \frac{-fg'}{f^2} = -\frac{g'}{f} \qquad \text{and} \qquad \kappa_2 = \frac{N}{G} = f''g' - f'g''$$

(d) The Gauss and mean curvature are

$$K = \kappa_1\kappa_2 = \frac{g'(f'g'' - f''g')}{f} \quad \text{and} \quad H = \frac{1}{2}(\kappa_1 + \kappa_2) = -\frac{g'}{2f} + \frac{1}{2}(f''g' - f'g'').$$

13.3. Let S be the surface in $\mathbb{R}^3$ defined by the equation

$$z = \frac{1}{1 + x^2 + y^2}.$$

Find the principal curvatures and the umbilic points (i.e., the points where the principal curvatures are the same). Give a sketch of the surface showing the regions of the surface where the Gauss curvature K is strictly positive and strictly negative.

Solution:

Consider S as a surface of revolution with the standard parametrization given by $\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$ with functions f and g to be determined. That $\boldsymbol{x}(u,v)$ is an element of the surface $S = \{\,(x,y,z) \mid z = 1/(1+x^2+y^2)\,\}$ means that

$$g(v) = \frac{1}{1 + f(v)^2}.$$

Choose e.g. $f(v) = v$ then $g(v) = 1/(1+v^2)$. As a parameter domain U we choose $U_1 = (-\pi,\pi) \times (0,\infty)$ and $U_2 = (0, 2\pi) \times (0,\infty)$.

Note: This parametrization covers all points on S *except* the point $(0,0,1) \in S$.

Calculating the coefficients of the first and second fundamental forms, we obtain

$$E = f^2 = v^2, \qquad F = 0, \qquad G = f'^2 + g'^2 = 1 + \frac{4}{v}^2 (1+v^2)^2$$

$$L = \frac{-fg'}{\sqrt{f'^2 + g'^2}}, \qquad M = 0, \qquad N = \frac{f''g' - f'g''}{\sqrt{f'^2 + g'^2}}$$

(see Example 9.13). In our concrete case, we have

$$f'(v) = 1, \quad f''(v) = 0, \quad g'(v) = \frac{-2v}{(1+v^2)^2}, \quad g''(v) = \frac{-2(1+v^2) + 2v(2v)2}{(1+v^2)^3} = \frac{2(3v^2-1)}{(1+v^2)^3}.$$

Since the parametrization is *principal* ($F = 0$ and $M = 0$), the principal curvatures are

$$\kappa_1 = \frac{L}{E} = \frac{-fg'}{f^2((f')^2 + (g')^2)^{1/2}} = -\frac{g'}{f((f')^2 + (g')^2)^{1/2}}, \quad \kappa_2 = \frac{N}{G} = \frac{(f''g' - f'g'')}{((f')^2 + (g')^2)^{3/2}},$$

which means here that

$$\kappa_1 = \frac{2}{(1+v^2)^2\left(1 + \frac{4v^2}{(1+v^2)^4}\right)^{1/2}} \quad \text{and} \quad \kappa_2 = -\frac{2(3v^2-1)}{(1+v^2)^3\left(1 + \frac{4v^2}{(1+v^2)^4}\right)^{3/2}}.$$

Now, a point is umbilic if $\kappa_1 = \kappa_2$ at this point, i.e., if

$$1 = -\frac{(3v^2 - 1)}{(1 + v^2)\Big(1 + \frac{4v^2}{(1+v^2)^4}\Big)},$$

or, equivalently, $(v > 0)$

$$\begin{aligned} 0 &= (1 + v^2)\Big(1 + \frac{4v^2}{(1 + v^2)^4}\Big) + (3v^2 - 1) \\ &= 4v^2 + \frac{4v^2}{(1 + v^2)^3} \end{aligned}$$

which has no solution if $v \neq 0$. Therefore, the surface has no umbilic point *on the points covered by the parametrization as surface of revolution*, i.e., the points $p \in S \setminus \{(0, 0, 1)\}$ are not umbilic.

What about the point $(0, 0, 1)$*?*

If we are just at the point $(0, 0, 1)$ (with parameter values $(u, v) = (0, 0)$ in the parametrization given by $\boldsymbol{x}(u, v) = (u, v, 1/(1 + u^2 + v^2))$), we obtain

$$f(x, y) = \frac{1}{1 + x^2 + y^2}, \qquad f_x(x, y) = \frac{-2x}{(1 + x^2 + y^2)^2}, \qquad f_y(x, y) = \frac{-2y}{(1 + x^2 + y^2)^2},$$

and

$$f_{xx}(x, y) = \frac{-2(1 + x^2 + y^2) + 2x2x2}{(1 + x^2 + y^2)^3} = \frac{-2(1 - 3x^2 + y^2)}{(1 + x^2 + y^2)^3}$$

and similarly

$$f_{xy}(x, y) = \frac{(-2)(-2x)(2y)}{(1 + x^2 + y^2)^3} = \frac{8xy}{(1 + x^2 + y^2)^3}, \qquad f_{yy}(x, y) = \frac{-2(1 + x^2 - 3y^2)}{(1 + x^2 + y^2)^3}.$$

Hence, we obtain for the coefficients of the first and second fundamental form at $(0, 0)$ the expressions

$$E(0, 0) = 1 + f_x(0, 0) = 1, \qquad F(0, 0) = f_x(0, 0)f_y(0, 0) = 0, \qquad G(0, 0) = 1 + f_y(0, 0) = 1.$$

Denote $D = 1 + f_x^2(0, 0) + f_y^2(0, 0) = 1$, then

$$L(0, 0) = \frac{f_{xx}(0, 0)}{D} = -2, \qquad M(0, 0) = \frac{f_{xy}(0, 0)}{D} = 0, \qquad N(0, 0) = \frac{f_{yy}(0, 0)}{D} = -2.$$

Therefore, the Gauss and mean curvatures at the parameter value $(0, 0)$ are

$$K = \frac{LN - M^2}{EG - F^2} = 4, \qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{-2 - 2}{2} = -2,$$

so that the principal curvatures are the roots of

$$\kappa^2 - 2H\kappa + K = 0, \qquad \text{or} \qquad \kappa^2 + 4 + 4 = (\kappa + 2)^2 = 0,$$

i.e., $\kappa_1 = \kappa_2 = -2$.

Therefore, $(0, 0, 1)$ is the only umbilic point of the surface (as one might already guess from the rotational symmetry of the surface).

One could start with this parametrization (as a graph) right from the beginning, but it seems that the formulas for the two principal curvaturs become much more complicated than as for a surface of revolution.

13.4. $(*)$ **The pseudosphere**

The pseudosphere is the surface of revolution obtained by rotating the tractrix with parametrization $\boldsymbol{\alpha}(s) = (1/\cosh s, 0, s - \tanh s)$ around the z-axis. Prove that the pseudosphere has constant Gauss curvature $K = -1$.

Solution:

Calculating the coefficients of the first and second fundamental forms, we obtain

$$E = f^2, \qquad F = 0, \qquad G = f'^2 + g'^2$$
$$L = \frac{-fg'}{\sqrt{f'^2 + g'^2}}, \qquad M = 0, \qquad N = \frac{f''g' - f'g''}{\sqrt{f'^2 + g'^2}}$$

(see Example 9.13). Let us assume that $v > 0$ (the surface for negative values v is just the mirror image w.r.t. the xy-plane).

In our case, we have

$$f(v) = \frac{1}{\cosh v}, \qquad f'(v) = -\frac{\sinh v}{\cosh^2 v}, \qquad f''(v) = -\frac{\cosh^2 v - 2\sinh^2 v}{\cosh^3 v} = \frac{\cosh^2 v - 2}{\cosh^3 v},$$

and

$$g(v) = v - \tanh v, \qquad g'(v) = 1 - \frac{1}{\cosh^2 v} = \frac{\cosh^2 v - 1}{\cosh^2 v} = \frac{\sinh^2 v}{\cosh^2 v}, \qquad g''(v) = \frac{2\sinh v}{\cosh^3 v}$$

Moreover, we have

$$f'(v)^2 + g'(v)^2 = \frac{\sinh^2 v + \sinh^4 v}{\cosh^4 v} = \frac{\sinh^2 v(1 + \sinh^2 v)}{\cosh^4 v} = \frac{\sinh^2 v \cosh^2 v}{\cosh^4 v} = \frac{\sinh^2 v}{\cosh^2 v} = \tanh^2 v$$

so that

$$E = \frac{1}{\cosh^2 v}, \qquad F = 0, \qquad G = \tanh^2 v$$
$$L = \frac{-\tanh^2 v/\cosh v}{\tanh v} = -\frac{\sinh v}{\cosh^2 v}, \qquad M = 0,$$
$$N = \frac{f''g' - f'g''}{\sqrt{f'^2 + g'^2}} = \frac{(\cosh^2 v - 2)\tanh^2 v/\cosh^3 v + 2\sinh^2 v/\cosh^5 v}{\tanh v} = \frac{\sinh v}{\cosh^2 v}$$

Since the parametrization is *principal* ($F = 0$ and $M = 0$), the principal curvatures are

$$\kappa_1 = \frac{L}{E} = -\frac{\sinh v}{\cosh^2 v \cosh^{-2} v} = -\sinh v,$$
$$\kappa_2 = \frac{N}{G} = \frac{\sinh v}{\cosh^2 v \tanh^2 v} = \frac{1}{\sinh v},$$

hence the Gauss curvature is $K = \kappa_1\kappa_2 = -1$, as desired.

14.1. Let S be the surface given by the graph of the function $f\colon U \longrightarrow \mathbb{R}$ ($U \subset \mathbb{R}^2$ open). Calculate the Gauss and mean curvature of S in terms of f and its derivatives.

Solution: We choose the standard parametrization for a graph of a function, i.e.,

$$\boldsymbol{x}\colon U \longrightarrow S, \qquad \boldsymbol{x}(u,v) = (u, v, f(u,v)),$$

where $S = \{\, (u, v, f(u,v)) \mid (u,v) \in U \,\}$. Then we have

$$\begin{aligned} \boldsymbol{x}_u &= (1,0,f_x), & \boldsymbol{x}_v &= (0,1,f_y), & \boldsymbol{x}_u \times \boldsymbol{x}_v &= (-f_x, -f_y, 1), \\ \boldsymbol{x}_{uu} &= (0,0,f_{xx}), & \boldsymbol{x}_{uv} &= (0,0,f_{xy}), & \boldsymbol{x}_{vv} &= (0,0,f_{yy}). \end{aligned}$$

From this, we see that the normal vector is

$$\boldsymbol{N} = \frac{1}{D}, \qquad D = \sqrt{1 + f_x^2 + f_y^2}$$

and we easily see that

$$\begin{aligned} E &= \boldsymbol{x}_u \cdot \boldsymbol{x}_u = 1 + f_x^2, & F &= \boldsymbol{x}_u \cdot \boldsymbol{x}_v = f_x f_y, & G &= \boldsymbol{x}_v \cdot \boldsymbol{x}_v = 1 + f_y^2, \\ L &= \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = \frac{f_{xx}}{D}, & M &= \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = \frac{f_{xy}}{D}, & N &= \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = \frac{f_{yy}}{D}. \end{aligned}$$

Note that we have

$$EG - F^2 = (1 + f_x^2)(1 + f_y^2) - f_x^2 f_y^2 = 1 + f_x^2 + f_y^2 = \|\boldsymbol{x}_u \times \boldsymbol{x}_v\|^2 = D^2.$$

(observe that the equality $EG - F^2 = \|\boldsymbol{x}_u \times \boldsymbol{x}_v\|^2$ is always true, what is the geometrical meaning of this?) Now, the Gauss curvature is given by

$$K = \frac{LN - M^2}{EG - F^2} = \frac{f_{xx} f_{yy} - f_{xy}^2}{D^4} = \frac{\det H(f)}{D^4}, \quad \text{where} \quad H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

is the Hessian matrix of f. Moreover, the mean curvature is given by

$$H = \frac{EN - 2FM + GL}{EG - F^2} = \frac{(1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} + (f_y^2 + 1) f_{xx}}{D^3}.$$

14.2. (∗) **Enneper's surface**

Consider the surface in $\mathbb{R}^3$ parametrized by

$$\boldsymbol{x}(u,v) = \Big(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\Big), \qquad (u,v) \in \mathbb{R}^2.$$

Show that

(a) the coefficients of the first and second fundamental forms are given by

$$E(u,v) = G(u,v) = (1 + u^2 + v^2)^2,\ F(u,v) = 0 \qquad \text{and} \qquad L = 2,\ M = 0,\ N = -2;$$

(b) the principal curvatures at $p = \boldsymbol{x}(u,v)$ are given by

$$\kappa_1(p) = \frac{2}{(1 + u^2 + v^2)^2}, \qquad \kappa_2(p) = -\frac{2}{(1 + u^2 + v^2)^2}.$$

Solution:

(a) We have

$$\boldsymbol{x}_u(u,v) = (1-u^2+v^2, 2uv, 2u), \qquad \boldsymbol{x}_v(u,v) = (2uv, 1+u^2-v^2, -2v)$$

so that the coefficients of the first fundamental form are

$$\begin{aligned} E(u,v) &= (1-u^2+v^2)^2 + 4uv^2 + 4u^2 = (1+u^2+v^2)^2, \\ F(u,v) &= 2uv(1-u^2+v^2) + 2uv(1+u^2-v^2) - 4uv = 0 \\ G(u,v) &= 4u^2v^2 + (1+u^2-v^2)^2 + 4v^2 = (1+u^2+v^2)^2 \end{aligned}$$

as desired. Moreover, we have

$$\boldsymbol{x}_{uu}(u,v) = (-2u, 2v, 2), \qquad \boldsymbol{x}_{uv}(u,v) = (2v, 2u, 0), \qquad \boldsymbol{x}_{vv}(u,v) = (2u, -2v, -2)$$

and

$$\begin{aligned} \boldsymbol{x}_u(u,v) \times \boldsymbol{x}_v(u,v) &= \begin{pmatrix} 1-u^2+v^2 \\ 2uv \\ 2u \end{pmatrix} \times \begin{pmatrix} 1+u^2-v^2 \\ 2uv \\ 2v \end{pmatrix} \\ &= \begin{pmatrix} -2u(1+u^2+v^2) \\ 2v(1+u^2+v^2) \\ (1-u^2-v^2)(1+u^2+v^2) \end{pmatrix} = (1+u^2+v^2) \begin{pmatrix} -2u \\ 2v \\ 1-u^2-v^2 \end{pmatrix} \end{aligned}$$

and $\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|^2 = EG - F^2 = (1+u^2+v^2)^4$, so that the normal vector is

$$\boldsymbol{N}(\boldsymbol{x}(u,v)) = \frac{1}{1+u^2+v^2} \begin{pmatrix} -2u \\ 2v \\ 1-u^2-v^2 \end{pmatrix}.$$

In particular, the coefficients of the second fundamental form are

$$\begin{aligned} L(u,v) &= \boldsymbol{x}_{uu} \cdot \boldsymbol{N}(\boldsymbol{x}(u,v)) = \frac{4u^2+4v^2+2(1-u^2-v^2)}{1+u^2+v^2} = 2 \\ M(u,v) &= \boldsymbol{x}_{uv} \cdot \boldsymbol{N}(\boldsymbol{x}(u,v)) = \frac{-4uv+4uv}{1+u^2+v^2} = 0 \\ N(u,v) &= \boldsymbol{x}_{uu} \cdot \boldsymbol{N}(\boldsymbol{x}(u,v)) = \frac{-4u^2-4v^2-2(1-u^2-v^2)}{1+u^2+v^2} = -2 \end{aligned}$$

again as desired.

(b) Let us first find the Gauss and mean curvature:

$$\begin{aligned} K &= \frac{LN-M^2}{EG-F^2} = \frac{-4}{(1+u^2+v^2)^4} \quad \text{and} \\ H &= \frac{EN-2FM+GL}{EG-F^2} = \frac{(-2+2)(1+u^2+v^2)^2}{(1+u^2+v^2)^4} = 0 \end{aligned}$$

hence the principal curvatures are the solutions of $\kappa^2 - 2H\kappa + K = 0$, i.e., of

$$\kappa^2 = \frac{4}{(1+u^2+v^2)^4}, \qquad \text{or} \qquad \kappa = \pm\frac{2}{(1+u^2+v^2)^2},$$

as desired.

Remark. Note that the mean curvature of the Enneper surface S vanishes, so it is a minimal surface.

14.3. If S is a surface in $\mathbb{R}^3$ then a *parallel surface* to S is a surface $\widetilde{S}$ given by a local parametrization of the form

$$\boldsymbol{y}(u,v) = \boldsymbol{x}(u,v) + a\boldsymbol{N}(u,v), \qquad (u,v) \in U,$$

where $\boldsymbol{x}\colon U \longrightarrow S$ is a local parametrization of S, $\boldsymbol{N}\colon U \longrightarrow S^2$ the Gauss map in that parametrization, and a is some given constant.

(a) Show that

$$\boldsymbol{y}_u \times \boldsymbol{y}_v = (1 - 2Ha + Ka^2)\, \boldsymbol{x}_u \times \boldsymbol{x}_v,$$

where H and K are the mean and Gauss curvatures of S.

(b) Assuming that $1-2Ha+Ka^2$ is never zero on S, show that the Gauss curvature $\widetilde{K}$ and mean curvature $\widetilde{H}$ of $\widetilde{S}$ are given by

$$\widetilde{K} = \frac{K}{1 - 2Ha + Ka^2}, \qquad \widetilde{H} = \frac{H - Ka}{1 - 2Ha + Ka^2}.$$

(c) If S has constant mean curvature $H \equiv c \neq 0$ and the Gauss curvature K is nowhere vanishing, show that the parallel surface given by $a = 1/(2c)$ has constant Gauss curvature $4c^2$.

Solution:

(a) First, note that

$$\boldsymbol{y}_u = \boldsymbol{x}_u + a\boldsymbol{N}_u, \qquad \text{and} \qquad \boldsymbol{y}_v = \boldsymbol{x}_u + a\boldsymbol{N}_v.$$

In order to express $\boldsymbol{y}_u \times \boldsymbol{y}_v$ in the desired form, it is helpful to express the derivative with respect to the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$:

$$\begin{aligned} -\boldsymbol{N}_u &= -(\boldsymbol{N} \circ x)_u = -d_p\boldsymbol{N}(\boldsymbol{x}_u) = A\boldsymbol{x}_u + B\boldsymbol{x}_v \quad \text{and} \\ -\boldsymbol{N}_v &= -(\boldsymbol{N} \circ x)_v = -d_p\boldsymbol{N}(\boldsymbol{x}_v) = C\boldsymbol{x}_u + D\boldsymbol{x}_v \end{aligned}$$

This is useful as we can express easily the Gauss and mean curvatures as the determinant and trace in terms of these coefficients as

$$K = AD - BC \qquad \text{and} \qquad H = \frac{A+D}{2}.$$

Now,

$$\begin{aligned} \boldsymbol{y}_u &= \boldsymbol{x}_u + a\boldsymbol{N}_u = \boldsymbol{x}_u + a(\boldsymbol{N} \circ \boldsymbol{x})_u = (1 - aA)\boldsymbol{x}_u - aB\boldsymbol{x}_v \quad \text{and} \\ \boldsymbol{y}_v &= \boldsymbol{x}_v + a\boldsymbol{N}_v = \boldsymbol{x}_v + a(\boldsymbol{N} \circ \boldsymbol{x})_v = -aC\boldsymbol{x}_u + (1 - aD)\boldsymbol{x}_v \end{aligned}$$

and therefore

$$\begin{aligned} \boldsymbol{y}_u \times \boldsymbol{y}_v &= \big((1 - aA)\boldsymbol{x}_u - aB\boldsymbol{x}_v\big) \times \big(-aC\boldsymbol{x}_u + (1 - aD)\boldsymbol{x}_v\big) \\ &= \big((1 - aA)(1 - aD) - a^2BC\big)\boldsymbol{x}_u \times \boldsymbol{x}_v \\ &= \big(1 - a(A + D) + a^2(AD - BC)\big)\boldsymbol{x}_u \times \boldsymbol{x}_v \\ &= \underbrace{(1 - 2Ha + Ka^2)}_{=:P}\, \boldsymbol{x}_u \times \boldsymbol{x}_v \end{aligned}$$

using the antisymmetry of the vector product ($\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$ and $\boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0}$), and we obtain the desired formula.

(b) If $P := 1 - 2Ha + Ka^2 \neq 0$, then $\boldsymbol{y}_u \times \boldsymbol{y}_v$ is not vanishing, the normal vectors of S and $\widetilde{S}$ fulfil

$$\widetilde{\boldsymbol{N}} \circ \boldsymbol{y} = \boldsymbol{N} \circ \boldsymbol{x},$$

as $\boldsymbol{y}_u \times \boldsymbol{y}_v$ and $\boldsymbol{x}_u \times \boldsymbol{x}_v$ point in the same direction by the first part and the condition on $1-2Ha+Ka^2$.

Remark. Be careful with the statement $\widetilde{\boldsymbol{N}} = \boldsymbol{N}$, as the parametrisation is lost in this espression. This becomes important when taking derivatives (see below).

Let us use the same trick as for the surface S also for $\widetilde{S}$:

$$\begin{aligned} -\widetilde{\boldsymbol{N}}_u &= -(\widetilde{\boldsymbol{N}} \circ y)_u = -d_p\widetilde{\boldsymbol{N}}(\boldsymbol{y}_u) = \widetilde{A}\boldsymbol{y}_u + \widetilde{B}\boldsymbol{y}_v \quad \text{and} \\ -\widetilde{\boldsymbol{N}}_v &= -(\widetilde{\boldsymbol{N}} \circ y)_v = -d_p\widetilde{\boldsymbol{N}}(\boldsymbol{y}_v) = \widetilde{C}\boldsymbol{y}_u + \widetilde{D}\boldsymbol{y}_v. \end{aligned}$$

Similarly as above, we have

$$\widetilde{K} = \widetilde{A}\widetilde{D} - \widetilde{B}\widetilde{C} \qquad \text{and} \qquad \widetilde{H} = \frac{\widetilde{A} + \widetilde{D}}{2}.$$

Taking the derivative of the equation $\widetilde{\boldsymbol{N}} \circ \boldsymbol{y} = \boldsymbol{N} \circ \boldsymbol{x}$ and combining the previous results gives

$$\begin{aligned} A\boldsymbol{x}_u + B\boldsymbol{x}_v = -(\boldsymbol{N} \circ \boldsymbol{x})_u &= -(\widetilde{\boldsymbol{N}} \circ \boldsymbol{y})_u \\ &= \widetilde{A}\boldsymbol{y}_u + \widetilde{B}\boldsymbol{y}_v \\ &= \widetilde{A}\big((1-aA)\boldsymbol{x}_u - aB\boldsymbol{x}_v\big) + \widetilde{B}\big(-aC\boldsymbol{x}_u + (1-aD)\boldsymbol{x}_v\big) \\ &= \big(\widetilde{A}(1-aA) - \widetilde{B}aC\big)\boldsymbol{x}_u + \big(-\widetilde{A}aB + \widetilde{B}(1-aD)\big)\boldsymbol{x}_v. \end{aligned}$$

Comparing the coefficients gives the linear system

$$\begin{pmatrix} 1-aA & -aC \\ -aB & 1-aD \end{pmatrix} \begin{pmatrix} \widetilde{A} \\ \widetilde{B} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

for $(\widetilde{A}, \widetilde{B})$. The determinant of the coefficient matrix is

$$(1-aA)(1-aD) - a^2BC = 1 - (A+D)a + (AD-BC)a^2 = 1 - 2Ha + Ka^2 = P \neq 0,$$

so that we can take the inverse and obtain

$$\begin{pmatrix} \widetilde{A} \\ \widetilde{B} \end{pmatrix} = \frac{1}{P}\begin{pmatrix} 1-aD & aC \\ aB & 1-aA \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{P}\begin{pmatrix} (1-aD)A + aCB \\ aBA + (1-aA)B \end{pmatrix} = \frac{1}{P}\begin{pmatrix} A - aK \\ B \end{pmatrix}.$$

Similarly, we have (taking the derivative w.r.t. v) that

$$\begin{aligned} C\boldsymbol{x}_u + D\boldsymbol{x}_v = -(\boldsymbol{N} \circ \boldsymbol{x})_v &= -(\widetilde{\boldsymbol{N}} \circ \boldsymbol{y})_v \\ &= \widetilde{C}\boldsymbol{y}_u + \widetilde{D}\boldsymbol{y}_v \\ &= \widetilde{C}\big((1-aA)\boldsymbol{x}_u - aB\boldsymbol{x}_v\big) + \widetilde{D}\big(-aC\boldsymbol{x}_u + (1-aD)\boldsymbol{x}_v\big) \\ &= \big(\widetilde{C}(1-aA) - \widetilde{D}aC\big)\boldsymbol{x}_u + \big(-\widetilde{C}aB + \widetilde{D}(1-aD)\big)\boldsymbol{x}_v. \end{aligned}$$

Comparing the coefficients gives the linear system

$$\begin{pmatrix} 1-aA & -aC \\ -aB & 1-aD \end{pmatrix} \begin{pmatrix} \widetilde{C} \\ \widetilde{D} \end{pmatrix} = \begin{pmatrix} C \\ D \end{pmatrix}$$

for $(\widetilde{B}, \widetilde{D})$, and as above, we obtain

$$\begin{pmatrix} \widetilde{C} \\ \widetilde{D} \end{pmatrix} = \frac{1}{P} \begin{pmatrix} 1-aD & aC \\ aB & 1-aA \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{P} \begin{pmatrix} (1-aD)C + aCD \\ aBC + (1-aA)D \end{pmatrix} = \frac{1}{P} \begin{pmatrix} C \\ D - aK \end{pmatrix}.$$

Now, we have

$$\widetilde{H} = \frac{1}{2}(\widetilde{A} + \widetilde{D}) = \frac{1}{2P}(A - aK + D - aK) = \frac{1}{P}(H - aK) = \frac{H - aK}{1 - 2aH + a^2K}$$

and

$$\begin{aligned} \widetilde{K} = \widetilde{A}\widetilde{D} - \widetilde{B}\widetilde{C} &= \frac{1}{P^2}\big((A - aK)(D - aK) - BC\big) \\ &= \frac{1}{P^2}\big(\underbrace{AD - BC}_{=K} - a(A + D)K + a^2K\big) \\ &= \frac{K(1 - 2aH + a^2K)}{(1 - 2aH + a^2K)^2} = \frac{K}{1 - 2aH + a^2K} \end{aligned}$$

as claimed.

(c) If S has constant mean curvature $H = c \neq 0$ and $K \neq 0$, then

$$\widetilde{K} = \frac{K}{1 - 2aH + a^2K} = \frac{K}{1 - 2c/2c + K/4c^2} = \frac{4c^2K}{K} = 4c^2$$

(and we have $P = 1 - 2aH + a^2K = K/4c^2 \neq 0$ as $K \neq 0$).

14.4. Let f be a smooth real-valued function defined on a connected open subset U of $\mathbb{R}^2$.

(a) Show that the graph S of f is a *minimal surface* in $\mathbb{R}^3$ (i.e., its mean curvature H vanishes) if and only if

$$f_{yy}(1 + f_x^2) - 2f_xf_yf_{xy} + f_{xx}(1 + f_y^2) = 0.$$

(b) Deduce that if $f(x, y) = g(x)$ then S is minimal if and only if S is a plane with normal vector parallel to the (x, z)-plane but not parallel to the x-axis.

(c) If $f(x, y) = g(x) + h(y)$, find the most general form of f in order for S to be minimal. *Hint: Use separation of variables*

Solution:

(a) Let us take the formulae for the mean curvature of a surface which is a graph of a function from Exercise 4.1 (feel free to repeat the calculations, it is a good exercise). We have

$$H = \frac{EN - 2FM + GL}{EG - F^2} = \frac{(1 + f_x^2)f_{yy} - 2f_xf_yf_{xy} + (f_y^2 + 1)f_{xx}}{D^3}$$

where $D = (1 + f_x^2 + f_y^2)^{1/2}$. In particular, a surface is a minimal surface iff

$$(1 + f_x^2)f_{yy} - 2f_xf_yf_{xy} + (f_y^2 + 1)f_{xx} = 0,$$

as desired.

(b) If $f(x,y) = g(x)$, then $f_x = g'$, $f_y = 0$, and the equation $H = 0$ becomes just $g'' = 0$ (only the third summand is non-zero). In particular, $g(x) = ax + b$ for some constants $a, b \in \mathbb{R}$, i.e., f is the graph of a plane, and the normal vector of this plane is proportional to $(-a, 0, 1)$, i.e., parallel to the (x,z)-plane, but not to the x-axis (as the z-component is never 0).

(c) If $f(x,y) = g(x) + h(y)$, we obtain

$$f_x = g', \qquad f_y = h', \qquad f_{xx} = g'', \qquad f_{xy} = 0, \qquad f_{yy} = h'',$$

so that the equation $H = 0$ becomes

$$(1 + g'^2)h''(h'^2 + 1)g'' = 0, \quad \text{i.e.} \quad \frac{g''}{1 + g'^2} = -\frac{h''}{h'^2 + 1}$$

(separation of variables). Now, since the LHS depends on x only, while the RHS depends on y only, we have

$$\frac{g''}{g'^2 + 1} = c_0$$

for some constant c_0. Integrating gives (substituting $s = g'(x)$, i.e., $\mathrm{d}s = g''(x)\,\mathrm{d}x$)

$$\int \frac{1}{s^2 + 1}\,\mathrm{d}s = c_0 x + c_1, \quad \text{i.e.} \quad \arctan g'(x) = c_0 x + c_1 \quad \text{or} \quad g'(x) = \tan(c_0 x + c_1).$$

Integrating gives $g(x) = -\log|\cos(c_0 x + c_1)|/c_0 + c_2$.
Similarly, $h(y) = \log|\cos(-c_0 y + c_3)|/c_0 + c_4$. So the most general form of f is

$$\begin{aligned} f(x,y) &= \frac{1}{c_0}\log|\cos(-c_0 x + c_3)| - \frac{1}{c_0}\log|\cos(c_0 y + c_1)|/c_0 + c_5 \\ &= \frac{1}{c_0}\log\left|\frac{\cos(-c_0 x + c_3)}{\cos(c_0 y + c_1)}\right| + c_5 \end{aligned}$$

where c_0, c_1, c_3, c_5 are constants.