Durham University
Anna Felikson

Differential Geometry 30.01.2020

Homework 13-14
Starred problems due on Thursday, 13 February.

Weingarten map, Gauss, mean and principal curvatures - 1

13.1. A local parametrization $\boldsymbol{x}$ of a surface S in $\mathbb{R}^3$ is called *orthogonal* provided $F = 0$ (so $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are orthogonal at each point). It is called *principal* if $F = 0$ and $M = 0$, where E, F, G (resp. L, M, N) are the coefficients of the first (resp. second) fundamental form.

(a) Let $\boldsymbol{x}$ be an *orthogonal* parametrization. Show that, at any point $p = \boldsymbol{x}(u, v)$ on S,

$$-d\boldsymbol{N}_p(\boldsymbol{x}_u) = \frac{L}{E}\boldsymbol{x}_u + \frac{M}{G}\boldsymbol{x}_v, \qquad -d\boldsymbol{N}_p(\boldsymbol{x}_v) = \frac{M}{E}\boldsymbol{x}_u + \frac{N}{G}\boldsymbol{x}_v,$$

where $\boldsymbol{N}$ denotes the Gauss map.

(b) Assume now that the parametrization is *principal.* Show that $\kappa_1 = L/E$ and $\kappa_2 = N/G$ are the principal curvatures. Calculate the Gauss and mean curvature in terms of E, G, L, N. Determine the principal directions.

13.2. Calculation of the Weingarten map directly for surfaces of revolution

Let $f\colon J \longrightarrow (0, \infty)$ and $g\colon J \longrightarrow \mathbb{R}$ be smooth functions on some open interval J in $\mathbb{R}$ and let $\alpha\colon J \longrightarrow \mathbb{R}^3$ be a space curve given by $\alpha(v) = (f(v), 0, g(v))$. Assume that this curve is parametrized by arc length. Let S be the surface of revolution obtained by rotating α around the z-axis.

(a) Find suitable parametrizations $\boldsymbol{x}\colon U_i \longrightarrow S$ of S and determine parameter domains U_1 and U_2 covering the whole surface S. Calculate the normal vector $\boldsymbol{N}$ at $\boldsymbol{x}(u, v)$

(b) Express $a, b, c, d \in \mathbb{R}$ in $-d\boldsymbol{N}_p(\boldsymbol{x}_u) = a\boldsymbol{x}_u + b\boldsymbol{x}_v$ and $-d\boldsymbol{N}_p(\boldsymbol{x}_v) = c\boldsymbol{x}_u + d\boldsymbol{x}_v$ in terms of f and g.

(c) Calculate the principal directions and principal curvatures.

(d) Calculate the Gauss and mean curvatures.

(e) Compare your results with Example 9.13 from the lectures.

13.3. Let S be the surface in $\mathbb{R}^3$ defined by the equation

$$z = \frac{1}{1 + x^2 + y^2}.$$

Find the principal curvatures and the umbilic points (i.e., the points where the principal curvatures are the same). Give a sketch of the surface showing the regions of the surface where the Gauss curvature K is strictly positive and strictly negative.

13.4. $(*)$ **The pseudosphere**

The pseudosphere is the surface of revolution obtained by rotating the tractrix with parametrization $\boldsymbol{\alpha}(s) = (1/\cosh s, 0, s - \tanh s)$ around the z-axis. Prove that the pseudosphere has constant Gauss curvature $K = -1$.

Weingarten map, Gauss, mean and principal curvatures - 2

14.1. Let S be the surface given by the graph of the function $f\colon U \longrightarrow \mathbb{R}$ ($U \subset \mathbb{R}^2$ open). Calculate the Gauss and mean curvature of S in terms of f and its derivatives.

14.2. ($*$) **Enneper's surface**

Consider the surface in $\mathbb{R}^3$ parametrized by

$$\boldsymbol{x}(u,v) = \Big(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\Big), \qquad (u,v) \in \mathbb{R}^2.$$

Show that

(a) the coefficients of the first and second fundamental forms are given by

$$E(u,v) = G(u,v) = (1+u^2+v^2)^2,\ F(u,v) = 0 \qquad \text{and} \qquad L = 2,\ M = 0,\ N = -2;$$

(b) the principal curvatures at $p = \boldsymbol{x}(u,v)$ are given by

$$\kappa_1(p) = \frac{2}{(1+u^2+v^2)^2}, \qquad \kappa_2(p) = -\frac{2}{(1+u^2+v^2)^2}.$$

14.3. If S is a surface in $\mathbb{R}^3$ then a *parallel surface* to S is a surface $\widetilde{S}$ given by a local parametrization of the form

$$\boldsymbol{y}(u,v) = \boldsymbol{x}(u,v) + a\boldsymbol{N}(u,v), \qquad (u,v) \in U,$$

where $\boldsymbol{x}\colon U \longrightarrow S$ is a local parametrization of S, $\boldsymbol{N}\colon U \longrightarrow S^2$ the Gauss map in that parametrization, and a is some given constant.

(a) Show that

$$\boldsymbol{y}_u \times \boldsymbol{y}_v = (1 - 2Ha + Ka^2)\, \boldsymbol{x}_u \times \boldsymbol{x}_v,$$

where H and K are the mean and Gauss curvatures of S.

(b) Assuming that $1 - 2Ha + Ka^2$ is never zero on S, show that the Gauss curvature $\widetilde{K}$ and mean curvature $\widetilde{H}$ of $\widetilde{S}$ are given by

$$\widetilde{K} = \frac{K}{1 - 2Ha + Ka^2}, \qquad \widetilde{H} = \frac{H - Ka}{1 - 2Ha + Ka^2}.$$

(c) If S has constant mean curvature $H \equiv c \neq 0$ and the Gauss curvature K is nowhere vanishing, show that the parallel surface given by $a = 1/(2c)$ has constant Gauss curvature $4c^2$.

14.4. Let f be a smooth real-valued function defined on a connected open subset U of $\mathbb{R}^2$.

(a) Show that the graph S of f is a *minimal surface* in $\mathbb{R}^3$ (i.e., its mean curvature H vanishes) if and only if

$$f_{yy}(1 + f_x^2) - 2f_x f_y f_{xy} + f_{xx}(1 + f_y^2) = 0.$$

(b) Deduce that if $f(x,y) = g(x)$ then S is minimal if and only if S is a plane with normal vector parallel to the (x,z)-plane but not parallel to the x-axis.

(c) If $f(x,y) = g(x) + h(y)$, find the most general form of f in order for S to be minimal.
Hint: Use separation of variables