Durham University
Anna Felikson

Solutions 15-16

15.1. Show that the Gauss curvature K of the surface of revolution locally parametrized by

$$\boldsymbol{x}(u,v) = (f(v)\cos(u), f(v)\sin(u), g(v)), \qquad (u,v) \in U,$$

(for some suitable parameter domain U) is given by

$$K = \frac{1}{2ff'}\Big(\frac{1}{1+(f'/g')^2}\Big)'.$$

If the generating curve is parametrized by arc length, show that $K = -f''/f$. Deduce Theorema Egregium in the latter case.

Solution: We have already calculated the coefficients of the first and second fundamental forms for a surface of revolution (see e.g. Example 9.13), so we just cite the result here again:

$$E = f^2, \qquad F = 0, \qquad G = f'^2 + g'^2$$
$$L = \frac{-fg'}{\sqrt{f'^2+g'^2}}, \qquad M = 0, \qquad N = \frac{f''g' - f'g''}{\sqrt{f'^2+g'^2}}.$$

If we assume now that $f(v) > 0$ everywhere, then we have

$$\kappa_1 = \frac{L}{E} = \frac{-g'}{f(f'^2+g'^2)^{1/2}}, \qquad \kappa_2 = \frac{N}{G} = \frac{f''g' - f'g''}{(f'^2+g'^2)^{3/2}}$$

(see Prop. 9.12), hence the Gauss curvature is

$$\begin{aligned} K = \kappa_1\kappa_2 = \frac{-g'(f''g'-f'g'')}{f(f'^2+g'^2)^2} &= \frac{-(f'/g')'}{fg'\big((f'/g')^2+1\big)^2} \\ &= \frac{-2(f'/g')(f'/g')'}{2f'f\big((f'/g')^2+1\big)^2} = \frac{1}{2ff'}\Big(\frac{1}{(f'/g')^2+1}\Big)' \end{aligned}$$

as desired (we have also implicitly assumed here that $f'(v) \neq 0 \neq g'(v)$). If the curve is parametrized by arc length, then $f'^2 + g'^2 = 1$, and

$$\kappa_1 = \frac{L}{E} = \frac{-g'}{f}, \qquad \kappa_2 = \frac{N}{G} = f''g' - f'g''.$$

Moreover, differentiating $f'^2 + g'^2 = 1$ gives $f'f'' + g'g'' = 0$, and we obtain

$$K = \kappa_1\kappa_2 = -\frac{g'(f''g'-f'g'')}{f} = -\frac{g'f''g' + f'(f'f'')}{f} = -\frac{(g'^2+f'^2)f''}{f} = -\frac{f''}{f}$$

as desired (this could also be obtained by simplifying the formula for K obtained above).

Now we can deduce Gauss' Theorema Egregium by expressing f in terms of the coefficients of the first fundamental form: $f = \sqrt{E}$. Then we have

$$f' = \frac{E_v}{2\sqrt{E}}, \qquad f'' = \frac{E_{vv}E - E_v^2/2}{2E^{3/2}},$$

so that

$$-f''/f = \frac{E_v^2}{4E^2} - \frac{E_v v}{2E} = -\frac{1}{2\sqrt{E}}\Big(\frac{E_v}{\sqrt{E}}\Big)_v$$

which is a special case ($G = 1$, i.e, $G_u = 0$) of the formula of Example 10.9.

15.2. Let $\boldsymbol{x}\colon U \longrightarrow S$ be a parametrization of a surface S for which $E = G = 1$ and $F = \cos(uv)$ (so that uv is the angle between the coordinate curves). Determine a suitable parameter domain U on which $\boldsymbol{x}(U)$ is a surface (i.e., where the coordinate curves are not tangential). Show that

$$K = -\frac{1}{\sin(uv)}.$$

Solution:

Suitable parameter domain: The tangent vectors $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are linearly dependent iff $F = \cos\vartheta = \pm 1$, where $\vartheta = uv$.

Another way to see this restriction is as follows: we have to assure that $\begin{pmatrix} 1 & \cos\vartheta \\ \cos\vartheta & 1 \end{pmatrix}$ with $\vartheta = uv$ is a positive definite matrix. Its determinant is $1 - \cos^2\vartheta$, and this is positive iff $\cos\vartheta \neq \pm 1$. Since its trace is always positive (the trace is 2), the matrix is positive definite iff $\cos\vartheta \neq \pm 1$.

So a maximal parameter domain could be

$$U := \{\, (u, v) \in \mathbb{R}^2 \mid uv \notin \pi\mathbb{Z} \,\},$$

or, if you prefer a connected domain, another choice could be

$$U := \{\, (u, v) \in \mathbb{R}^2 \mid 0 < u < \pi/v, \quad v > 0 \,\}.$$

(choosing just the component $0 < uv < \pi$).

The further calculations are similar to ones used in Example 10.7 (and in the proof of Theorema Egregium). We amend the order a bit to avoid computations with some zeros, and thus to save time in this way.

(a) *Step 1: Christoffel symbols* Γ_{ij}^k are functions defined by

$$\boldsymbol{x}_{uu} = \Gamma_{11}^1 \boldsymbol{x}_u + \Gamma_{11}^2 \boldsymbol{x}_v + L\boldsymbol{N} \tag{Γ1}$$

$$\boldsymbol{x}_{uv} = \Gamma_{12}^1 \boldsymbol{x}_u + \Gamma_{12}^2 \boldsymbol{x}_v + M\boldsymbol{N} \tag{Γ2}$$

$$\boldsymbol{x}_{vv} = \Gamma_{22}^1 \boldsymbol{x}_u + \Gamma_{22}^2 \boldsymbol{x}_v + N\boldsymbol{N} \tag{Γ3}$$

(and we have $\Gamma_{12}^k = \Gamma_{21}^k$ since $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$).
Before calculating Γ_{ij}^k in terms of E, F and G, let us first see what we need (to save some time).
But we also need the following: Express $\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u$ etc. in terms of E, F, G:

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u = \frac{1}{2}(\boldsymbol{x}_u \cdot \boldsymbol{x}_u)_u = \frac{1}{2}E_u \quad (= 0) \tag{1}$$

$$\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u = \frac{1}{2}(\boldsymbol{x}_u \cdot \boldsymbol{x}_u)_v = \frac{1}{2}E_v \quad (= 0) \tag{2}$$

$$\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v = \frac{1}{2}(\boldsymbol{x}_v \cdot \boldsymbol{x}_v)_u = \frac{1}{2}G_u \quad (= 0) \tag{3}$$

$$\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v = \frac{1}{2}(\boldsymbol{x}_v \cdot \boldsymbol{x}_v)_v = \frac{1}{2}G_v \quad (= 0) \tag{4}$$

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v = (\boldsymbol{x}_u \cdot \boldsymbol{x}_v)_u - \boldsymbol{x}_u \cdot \boldsymbol{x}_{uv} = F_u - \frac{1}{2}E_v \quad (= -v\sin(uv)) \tag{5}$$

$$\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u = (\boldsymbol{x}_v \cdot \boldsymbol{x}_u)_v - \boldsymbol{x}_v \cdot \boldsymbol{x}_{uv} = F_v - \frac{1}{2}G_u \quad (= -u\sin(uv)) \tag{6}$$

(the terms in parentheses correspond to our special case $E = G = 1$, $F = \cos(uv)$).
Multiplying the defining equations for Γ_{ij}^k by $\cdot \boldsymbol{x}_u$ and $\cdot \boldsymbol{x}_v$, we obtain equations

$$\begin{aligned} &E\Gamma_{11}^1 + F\Gamma_{11}^2 = \frac{1}{2}E_u, && E\Gamma_{12}^1 + F\Gamma_{12}^2 = \frac{1}{2}E_v, && E\Gamma_{22}^1 + F\Gamma_{22}^2 = F_v - \frac{1}{2}G_u, \\ &F\Gamma_{11}^1 + G\Gamma_{11}^2 = F_u - \frac{1}{2}E_v, && F\Gamma_{12}^1 + G\Gamma_{12}^2 = \frac{1}{2}G_u, && F\Gamma_{22}^1 + G\Gamma_{22}^2 = \frac{1}{2}G_v. \end{aligned}$$

Plugging in $E = G = 1$ and $F = \cos(uv)$, we obtain

$$\begin{aligned} &\Gamma_{11}^1 + \cos(uv)\Gamma_{11}^2 = 0, && \Gamma_{12}^1 + \cos(uv)\Gamma_{12}^2 = 0, && \Gamma_{22}^1 + \cos(uv)\Gamma_{22}^2 = -u\sin(uv), \\ &\cos(uv)\Gamma_{11}^1 + \Gamma_{11}^2 = -v\sin(uv), && \cos(uv)\Gamma_{12}^1 + \Gamma_{12}^2 = 0, && \cos(uv)\Gamma_{22}^1 + \Gamma_{22}^2 = 0. \end{aligned}$$

From this one *could* easily obtain that

$$\begin{aligned} &\Gamma_{11}^1 = \frac{v\cos(uv)}{\sin(uv)}, && \Gamma_{12}^1 = 0, && \Gamma_{22}^1 = -\frac{u}{\sin(uv)}, \\ &\Gamma_{11}^2 = -\frac{v}{\sin(uv)}, && \Gamma_{12}^2 = 0, && \Gamma_{22}^2 = \frac{u\cos(uv)}{\sin(uv)}. \end{aligned}$$

However, we will see now that we can avoid computations of Γ_{22}^1 and Γ_{22}^2.

(b) *Step 2: Calculate $LN - M^2$:*
From the equations above we have

$$\begin{aligned} LN &= (L\boldsymbol{N}) \cdot (N\boldsymbol{N}) \\ &= (\boldsymbol{x}_{uu} - \Gamma_{11}^1\boldsymbol{x}_u - \Gamma_{11}^2\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma_{22}^1\boldsymbol{x}_u - \Gamma_{22}^2\boldsymbol{x}_v) \\ &= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \Gamma_{22}^1 \underbrace{\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u}_{=0} - \Gamma_{22}^2 \underbrace{\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v}_{=-v\sin(uv)} - \Gamma_{11}^1 \underbrace{\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u}_{=-u\sin(uv)} - \Gamma_{11}^2 \underbrace{\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v}_{=0} \\ &\qquad + \Gamma_{11}^1\Gamma_{22}^1 E + (\Gamma_{11}^1\Gamma_{22}^2 + \Gamma_{11}^2\Gamma_{22}^1)F + \Gamma_{11}^2\Gamma_{22}^2 G \\ &= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} + (\Gamma_{22}^2 v + \Gamma_{11}^1 u)\sin(uv) \\ &\qquad + \Gamma_{11}^1\Gamma_{22}^1 + (\Gamma_{11}^1\Gamma_{22}^2 + \Gamma_{11}^2\Gamma_{22}^1)\cos(uv) + \Gamma_{11}^2\Gamma_{22}^2 \\ &= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} + \Gamma_{22}^2\big(\Gamma_{11}^1 + \Gamma_{11}^2 + v\sin(uv)\big) + \Gamma_{22}^1\big(\Gamma_{11}^1 + \Gamma_{11}^2\cos(uv)\big) + \Gamma_{11}^1 u\sin(uv). \end{aligned}$$

Note that due to the defining equations on Γ_{11}^1 and Γ_{11}^2 the first and second parentheses in the expression above vanish, i.e.,

$$LN = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} + \Gamma_{11}^1 u\sin(uv),$$

which means that all we needed is to calculate $\Gamma_{11}^1 = \frac{v\cos(uv)}{\sin(uv)}$.
Let us now calculate M^2. First we observe that the linear system involving Γ_{12}^1 and Γ_{12}^2 is homogeneous with non-zero determinant, so has a trivial solution only, i.e. $\Gamma_{12}^1 = 0 = \Gamma_{12}^2$. Hence,

$$\begin{aligned} M^2 &= (M\boldsymbol{N}) \cdot (M\boldsymbol{N}) \\ &= (\boldsymbol{x}_{uv} - \Gamma_{12}^1\boldsymbol{x}_u - \Gamma_{12}^2\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma_{12}^1\boldsymbol{x}_u - \Gamma_{12}^2\boldsymbol{x}_v) \\ &= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv}, \end{aligned}$$

so that

$$LN - M^2 = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} + \Gamma_{11}^1 u\sin(uv) = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} + uv\cos(uv).$$

Finally, recall that

$$\begin{aligned}\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_{vv}-\boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv} &= (\boldsymbol{x}_u\cdot\boldsymbol{x}_{vv})_u-(\boldsymbol{x}_u\cdot\boldsymbol{x}_{uv})_v\\ &= \left(F_v-G_u/2\right)_u-(E_v/2)_v\\ &= (-u\sin(uv))_u-0\\ &= -\sin(uv)-uv\cos(uv),\end{aligned}$$

so that finally,

$$\begin{aligned}LN-M^2 &= -\sin(uv)-uv\cos(uv)+uv\cos(uv)\\ &= -\sin(uv).\end{aligned}$$

(c) *Step 3: Calculate K:*
The Gauss curvature is

$$K=\frac{LN-M^2}{EG-F^2}=\frac{LN-M^2}{1-\cos^2(uv)}=\frac{-1}{\sin(uv)}$$

as desired.

15.3. $(*)$ If the coefficients of the first fundamental form of a surface S are given by

$$E=2+v^2,\qquad F=1,\qquad G=1,$$

show that the Gauss curvature of S is given by

$$K=-\frac{1}{(1+v^2)^2}.$$

Solution:

Calculations are similar to the previous exercise.

(a) *Step 1: Christoffel symbols Γ_{ij}^k.*
We have

$$\begin{aligned}E\Gamma_{11}^1+F\Gamma_{11}^2&=\frac{1}{2}E_u, & E\Gamma_{12}^1+F\Gamma_{12}^2&=\frac{1}{2}E_v, & E\Gamma_{22}^1+F\Gamma_{22}^2&=F_v-\frac{1}{2}G_u,\\ F\Gamma_{11}^1+G\Gamma_{11}^2&=F_u-\frac{1}{2}E_v, & F\Gamma_{12}^1+G\Gamma_{12}^2&=\frac{1}{2}G_u, & F\Gamma_{22}^1+G\Gamma_{22}^2&=\frac{1}{2}G_v.\end{aligned}$$

Plugging in $E=2+v^2$ and $F=G=1$, we obtain

$$\begin{aligned}(2+v^2)\Gamma_{11}^1+\Gamma_{11}^2&=0, & (2+v^2)\Gamma_{12}^1+\Gamma_{12}^2&=v, & (2+v^2)\Gamma_{22}^1+\Gamma_{22}^2&=0,\\ \Gamma_{11}^1+\Gamma_{11}^2&=-v, & \Gamma_{12}^1+\Gamma_{12}^2&=0, & \Gamma_{22}^1+\Gamma_{22}^2&=0.\end{aligned}$$

We see that the equations on Γ_{22}^1 and Γ_{22}^2 have only the trivial solution $\Gamma_{22}^1=\Gamma_{22}^2=0$. For the others, we obtain

$$\Gamma_{11}^1=\frac{v}{(1+v^2)},\qquad \Gamma_{11}^2=-\frac{2v+v^3}{1+v^2},\qquad \Gamma_{12}^1=\frac{v}{1+v^2},\qquad \Gamma_{12}^2=-\frac{v}{1+v^2}.$$

(b) *Step 2: Calculate $LN - M^2$:*
We have

$$\begin{aligned}
LN &= (L\boldsymbol{N}) \cdot (N\boldsymbol{N}) \\
&= (\boldsymbol{x}_{uu} - \Gamma_{11}^1 \boldsymbol{x}_u - \Gamma_{11}^2 \boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma_{22}^1 \boldsymbol{x}_u - \Gamma_{22}^2 \boldsymbol{x}_v) \\
&= (\boldsymbol{x}_{uu} - \Gamma_{11}^1 \boldsymbol{x}_u - \Gamma_{11}^2 \boldsymbol{x}_v) \cdot \boldsymbol{x}_{vv} \\
&= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \Gamma_{11}^1 \boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u - \Gamma_{11}^2 \boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v
\end{aligned}$$

So we only need $\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v = G_v/2 = 0$ and $\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u = F_v - G_u/2 = 0$ in our case here, hence we have

$$LN = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv}.$$

Similarly, for M^2 we obtain

$$\begin{aligned}
M^2 &= (M\boldsymbol{N}) \cdot (M\boldsymbol{N}) \\
&= (\boldsymbol{x}_{uv} - \Gamma_{12}^1 \boldsymbol{x}_u - \Gamma_{12}^2 \boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma_{22}^1 \boldsymbol{x}_u - \Gamma_{22}^2 \boldsymbol{x}_v) \\
&= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} - 2\Gamma_{12}^1 \underbrace{\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u}_{=E_v/2=v} - 2\Gamma_{12}^2 \underbrace{\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v}_{=G_u/2=0} \\
&\qquad + (\Gamma_{12}^1)^2 \underbrace{\boldsymbol{x}_u \cdot \boldsymbol{x}_u}_{=E=2+v^2} + 2\Gamma_{12}^1 \Gamma_{12}^2 \underbrace{\boldsymbol{x}_u \cdot \boldsymbol{x}_v}_{=F=1} + (\Gamma_{12}^2)^2 \underbrace{\boldsymbol{x}_v \cdot \boldsymbol{x}_v}_{=G=1} \\
&= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} - \frac{2v^2}{1+v^2} \\
&\qquad + \frac{v^2(2+v^2)}{(1+v^2)^2} - \frac{2v^2}{(1+v^2)^2} + \frac{v^2}{(1+v^2)^2} \\
&= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} - \frac{2v^2}{1+v^2} \\
&\qquad + \frac{v^2(1+v^2)}{(1+v^2)^2} \\
&= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} - \frac{v^2}{1+v^2}
\end{aligned}$$

Hence

$$LN - M^2 = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} + \frac{v^2}{1+v^2}.$$

Recall again that

$$\begin{aligned}
\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} &= (\boldsymbol{x}_u \cdot \boldsymbol{x}_{vv})_u - (\boldsymbol{x}_u \cdot \boldsymbol{x}_{uv})_v \\
&= \left(F_v - \frac{1}{2} G_u\right)_u - \left(\frac{1}{2} E_v\right)_v \\
&= -1,
\end{aligned}$$

so that finally

$$LN - M^2 = -1 + \frac{v^2}{1+v^2} = -\frac{1}{1+v^2}$$

(c) *Step 3: Calculate K:*
The Gauss curvature is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-1/(1+v^2)}{(2+v^2) - 1} = \frac{-1}{(1+v^2)^2}$$

as desired.

15.4. Let $\boldsymbol{x}$ be a local parametrization of a surface S such that $E = 1$, $F = 0$ and G is a function of u only. Show that

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{G_u}{2G}, \quad \Gamma^1_{22} = -\frac{G_u}{2}$$

and that all the other Christoffel symbols are zero. Hence show that the Gauss curvature K of S is given by

$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}.$$

Solution:

Again, the calculations are similar to the previous exercise.

(a) *Step 1: Christoffel symbols Γ^k_{ij}.*
Since $E = 1$, $F = 0$ and $G_v = 0$, we have

$$\Gamma^1_{11} = 0, \qquad \Gamma^1_{12} = 0, \qquad \Gamma^1_{22} = -\frac{1}{2}G_u,$$

$$G\Gamma^2_{11} = 0, \qquad G\Gamma^2_{12} = \frac{1}{2}G_u, \qquad G\Gamma^2_{22} = 0,$$

which implies

$$(\Gamma^2_{21} =)\Gamma^2_{12} = \frac{G_u}{2G} \qquad \text{and} \qquad \Gamma^1_{22} = -\frac{G_u}{2},$$

and all other Christoffel symbols are 0, as desired (note that G cannot vanish as the first fundamental form is positive definite).

(b) *Step 2: Calculate $LN - M^2$.*
We have

$$\begin{aligned}
LN &= (L\boldsymbol{N}) \cdot (N\boldsymbol{N}) \\
&= (\boldsymbol{x}_{uu} - \Gamma^1_{11}\boldsymbol{x}_u - \Gamma^2_{11}\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma^1_{22}\boldsymbol{x}_u - \Gamma^2_{22}\boldsymbol{x}_v) \\
&= \boldsymbol{x}_{uu} \cdot (\boldsymbol{x}_{vv} - \Gamma^1_{22}\boldsymbol{x}_u) \\
&= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \Gamma^1_{22} \underbrace{\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u}_{=E_u/2=0} \\
&= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv}.
\end{aligned}$$

Similarly, for M^2 we obtain

$$\begin{aligned}
M^2 &= (M\boldsymbol{N}) \cdot (M\boldsymbol{N}) \\
&= (\boldsymbol{x}_{uv} - \Gamma^1_{12}\boldsymbol{x}_u - \Gamma^2_{12}\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma^1_{22}\boldsymbol{x}_u - \Gamma^2_{22}\boldsymbol{x}_v) \\
&= (\boldsymbol{x}_{uv} - \Gamma^2_{12}\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma^2_{12}\boldsymbol{x}_v) \\
&= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} - 2\Gamma^2_{12} \underbrace{\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v}_{=G_u/2} + (\Gamma^2_{12})^2 \underbrace{\boldsymbol{x}_v \cdot \boldsymbol{x}_v}_{=G} \\
&= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} - \frac{(G_u)^2}{2G} + \frac{(G_u)^2}{4G} = \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} - \frac{(G_u)^2}{4G}
\end{aligned}$$

Hence,

$$LN - M^2 = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} + \frac{(G_u)^2}{4G}$$

Recall again that

$$\begin{aligned}\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_{vv}-\boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv} &= (\boldsymbol{x}_u\cdot\boldsymbol{x}_{vv})_u-(\boldsymbol{x}_u\cdot\boldsymbol{x}_{uv})_v\\ &= \Big(F_v-\frac{1}{2}G_u\Big)_u-\Big(\frac{1}{2}E_v\Big)_v\\ &= -\frac{1}{2}G_{uu},\end{aligned}$$

so that finally

$$LN-M^2=-\frac{1}{2}G_{uu}+\frac{(G_u)^2}{4G}.$$

(c) *Step 3: Calculate K.*
The Gauss curvature is

$$K=\frac{LN-M^2}{EG-F^2}=\frac{-G_{uu}/2+(G_u)^2/4G}{G}=-\frac{G_{uu}}{2G}+\frac{(G_u)^2}{4G^2}.$$

Finally, we have

$$\frac{(\sqrt{G})_{uu}}{\sqrt{G}}=\frac{1}{\sqrt{G}}\Big(\frac{G_u}{2\sqrt{G}}\Big)_u=\frac{1}{\sqrt{G}}\Big(\frac{G_{uu}}{2\sqrt{G}}-\frac{(G_u)^2}{4G^{3/2}}\Big)=\frac{G_{uu}}{2G}-\frac{(G_u)^2}{4G^2}=-K$$

so that we obtain the desired formula.

Remark. A particular example of such coefficients is given by a surface of revolution with a generating curve parametrized by arc length with u and v interchanged.

16.1. Let $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ be an orthonormal basis of T_pS consisting of eigenvectors of the Weingarten map $-d_p\boldsymbol{N}$ with corresponding eigenvalues κ_1, κ_2. If $\boldsymbol{e}=(\cos\vartheta)\boldsymbol{e}_1+(\sin\vartheta)\boldsymbol{e}_2$, show, that the normal curvature κ_{n} of a curve tangential to $\boldsymbol{e}$ is given by

$$\kappa_{\mathrm{n}}(\vartheta)=\kappa_1\cos^2\vartheta+\kappa_2\sin^2\vartheta.$$

Deduce that

$$\frac{1}{2\pi}\int_0^{2\pi}\kappa_{\mathrm{n}}(\vartheta)\,\mathrm{d}\vartheta=H,$$

where H denotes the mean curvature of S at p. (This justifies the term *mean curvature*).

Solution:

Note first that

$$I_p(\boldsymbol{e})=\|\boldsymbol{e}\|^2=\big\|(\cos\vartheta)\boldsymbol{e}_1+(\sin\vartheta)\boldsymbol{e}_2\big\|^2=\cos^2\vartheta+\sin^2\vartheta=1$$

by Pythagoras' Theorem (as $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are orthonormal). The normal curvature κ_{n} of a curve with tangent vector $\boldsymbol{e}$ at p is given by

$$\begin{aligned}\kappa_{\mathrm{n}}(\vartheta)=\frac{I\!I_p(\boldsymbol{e})}{I_p(\boldsymbol{e})}&=I\!I_p(\boldsymbol{e})\\ &=I\!I_p\big((\cos\vartheta)\boldsymbol{e}_1+(\sin\vartheta)\boldsymbol{e}_2\big)\\ &=-\big\langle d_p\boldsymbol{N}((\cos\vartheta)\boldsymbol{e}_1+(\sin\vartheta)\boldsymbol{e}_2),(\cos\vartheta)\boldsymbol{e}_1+(\sin\vartheta)\boldsymbol{e}_2\big\rangle\\ &=\big\langle\kappa_1(\cos\vartheta)\boldsymbol{e}_1+\kappa_2(\sin\vartheta)\boldsymbol{e}_2),(\cos\vartheta)\boldsymbol{e}_1+(\sin\vartheta)\boldsymbol{e}_2\big\rangle\\ &=\kappa_1\cos^2\vartheta+\kappa_2\sin^2\vartheta\end{aligned}$$

by Meusnier's theorem (first equality), the definition of the second fundamental form (fourth equality), the fact that $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are eigenvectors of $-d_p\boldsymbol{N}$ (fifth equality) and that $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are orthonormal (last equality). This shows the first formula.

For the second, just note that

$$\frac{1}{2\pi}\int_0^{2\pi}\kappa_{\mathrm{n}}(\vartheta)\,\mathrm{d}\vartheta = \frac{1}{2\pi}\int_0^{2\pi}(\kappa_1\cos^2\vartheta + \kappa_2\sin^2\vartheta)\,\mathrm{d}\vartheta = \frac{1}{2\pi}(\kappa_1\pi + \kappa_2\pi) = \frac{1}{2}(\kappa_1+\kappa_2) = H(p)$$

as $\int_0^{2\pi}\cos^2\vartheta\,\mathrm{d}\vartheta = \pi$ and similarly for the integral over $\sin^2\vartheta$.

16.2. Let $\boldsymbol{\alpha}$ be the curve defined by

$$\boldsymbol{\alpha}(t) = \varepsilon^t(\cos t, \sin t, 1) \qquad \text{for } t \in \mathbb{R}.$$

Observe that $\boldsymbol{\alpha}$ lies on the circular cone $S = \{\,(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=z^2\,\}$.

Show that the normal curvature of $\boldsymbol{\alpha}$ in S is inversely proportional to ε^t.

Solution:

Clearly,

$$(\varepsilon^t\cos t)^2 + (\varepsilon^t\sin t)^2 = (\varepsilon^t)^2,$$

so $\boldsymbol{\alpha}(t)\in S$ for all $t\in\mathbb{R}$. For further purposes, we also need

$$\boldsymbol{\alpha}'(t) = \varepsilon^t(\cos t - \sin t, \sin t + \cos t, 1).$$

Calculation of the normal curvature — reparametrization by arc length: Since $\|\boldsymbol{\alpha}'(t)\| = \sqrt{3}\varepsilon^t$ we set

$$s = \int_{-\infty}^{t}\sqrt{3}\varepsilon^u\,\mathrm{d}u = \sqrt{3}\varepsilon^t$$

so that $t = \log(s/\sqrt{3}) = \log s - (\log 3)/2$. Let us now call the reparametrized curve $\boldsymbol{\beta}$, i.e., we set

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(\log(s/\sqrt{3})) = \frac{s}{\sqrt{3}}\Big(\cos\log\frac{s}{\sqrt{3}}, \sin\log\frac{s}{\sqrt{3}}, 1\Big)$$

and therefore, we have

$$\begin{aligned}
\boldsymbol{\beta}'(s) &= \frac{1}{\sqrt{3}}\Big(\cos\log\frac{s}{\sqrt{3}} - \sin\log\frac{s}{\sqrt{3}}, \sin\log\frac{s}{\sqrt{3}} + \cos\log\frac{s}{\sqrt{3}}, 1\Big),\\
\boldsymbol{\beta}''(s) &= \frac{1}{s\sqrt{3}}\Big(-\sin\log\frac{s}{\sqrt{3}} - \cos\log\frac{s}{\sqrt{3}}, \cos\log\frac{s}{\sqrt{3}} - \sin\log\frac{s}{\sqrt{3}}, 0\Big)
\end{aligned}$$

How can we efficiently calculate the normal vector for a surface defined by an equation? At $p=(x,y,z)$, for a surface $S = \{\,(x,y,z)\mid f(x,y,z)=0\,\}$ we have (here with $f(x,y,z)=x^2+y^2-z^2$, hence $\nabla f(x,y,z) = 2(x,y,-z)$)

$$\boldsymbol{N}(p) = \frac{1}{\|\nabla f(p)\|}\nabla f(p) = \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,-z)\ \Big(= \frac{1}{\|p\|}(x,y,-z) \text{ here.}\Big)$$

— So there is no need to find a parametrization and then calculate $\boldsymbol{x}_u \times \boldsymbol{x}_v$ etc. —

Now, the normal curvature of the curve $\boldsymbol{\beta}$ (and hence $\boldsymbol{\alpha}$) is

$$\begin{aligned}
\kappa_{\mathrm{n},\boldsymbol{\beta}}(s) &= \boldsymbol{\beta}''(s) \cdot \boldsymbol{N}(\boldsymbol{\beta}(s)) \\
&= \frac{1}{s\sqrt{3}\|\boldsymbol{\beta}(s)\|}\bigg(\Big(-\sin\log\frac{s}{\sqrt{3}} - \cos\log\frac{s}{\sqrt{3}}\Big)\frac{s}{\sqrt{3}}\cos\log\frac{s}{\sqrt{3}} \\
&\qquad + \Big(\cos\log\frac{s}{\sqrt{3}} - \sin\log\frac{s}{\sqrt{3}}\Big)\frac{s}{\sqrt{3}}\sin\log\frac{s}{\sqrt{3}}\bigg) \\
&= \frac{-1}{3\|\boldsymbol{\beta}(s)\|}\Big(\cos^2\log\frac{s}{\sqrt{3}} + \sin^2\log\frac{s}{\sqrt{3}}\Big) = -\frac{1}{3\|\boldsymbol{\beta}(s)\|}
\end{aligned}$$

and since

$$\|\boldsymbol{\beta}(s)\|^2 = \frac{s^2}{3}\Big(\cos^2\log\frac{s}{\sqrt{3}} + \sin^2\log\frac{s}{\sqrt{3}} + 1\Big) = \frac{2s^2}{3},$$

we have $\kappa_{\mathrm{n},\boldsymbol{\beta}}(s) = -1/(3\sqrt{2s^2/3}) = -(\sqrt{3/2})/(3s)$ and finally

$$\kappa_{\mathrm{n},\boldsymbol{\alpha}}(t) = \kappa_{\mathrm{n},\boldsymbol{\beta}}(\sqrt{3}\varepsilon^t) = -\sqrt{\frac{3}{2}} \cdot \frac{1}{3\sqrt{3}\varepsilon^t} = -\frac{1}{3\sqrt{2}\varepsilon^t}$$

which is inversely proportional to ε^t as desired.

Alternative approach: Calculation of the normal curvature using a local parametrization. If we parametrize the surface S as a surface of revolution by

$$\boldsymbol{x}(u,v) = (v\cos u, v\sin u, v), \qquad (u,v) \in (-\pi,\pi) \times (0,\infty) \text{ or } (u,v) \in (0,2\pi) \times (0,\infty)$$

then $\boldsymbol{\alpha}$ is given in these parametrization as

$$\boldsymbol{\alpha}(t) = (\varepsilon^t\cos t, \varepsilon^t \sin t, \varepsilon^t)\boldsymbol{x}(u(t), v(t))$$

which means that

$$u(t) = t \qquad \text{and} \qquad v(t) = \varepsilon^t.$$

Now, the formula for the normal curvature of $\boldsymbol{\alpha}$ in a local parametrization is given by

$$\kappa_{\mathrm{n}} = \frac{(u')^2L + 2u'v'M + (v')^2N}{(u')^2E + 2u'v'F + (v')^2G},$$

so we need the coefficients of the first and second fundamental form. Since

$$\boldsymbol{x}_u = (-v\sin u, v\cos u, 0) \qquad \text{and} \qquad \boldsymbol{x}_v = (\cos u, \sin u, 1),$$

we have $\boldsymbol{x}_u \times \boldsymbol{x}_v = v(\cos u, \sin u, -1)$, hence

$$\boldsymbol{N} = \frac{1}{\sqrt{2}}(\cos u, \sin u, -1)$$

and $E(u,v) = v^2$, $F = 0$ and $G = 2$. Moreover,

$$\boldsymbol{x}_{uu} = (-v\cos u, -v\sin u, 0), \qquad \boldsymbol{x}_{uv} = (-\sin u, \cos u, 0), \qquad \boldsymbol{x}_{vv} = (0,0,0),$$

so that

$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = -\frac{v}{\sqrt{2}}, \qquad M = 0, \qquad N = 0.$$

Moreover, $u'(t) = 1$ and $v'(t) = \varepsilon^t$, so that finally

$$\kappa_{\mathrm{n}} = \frac{-(u')^2 v/\sqrt{2}}{(u')^2 v^2 + 2(v')^2} = \frac{-\varepsilon^t/\sqrt{2}}{\varepsilon^{2t} + 2\varepsilon^{2t}} = -\frac{1}{\varepsilon^t 3\sqrt{2}}$$

16.3. Show that an asymptotic curve can only exist in the hyperbolic or flat region $\{\, p \in S \mid K(p) \leq 0 \,\}$. (In other words, if a surface is elliptic everywhere, then there is no asymptotic curve.)

Solution:

A curve is an asymptotic curve iff $I\!I_{\boldsymbol{\alpha}(s)}(\alpha') = 0$. If $K(p) > 0$, then $LN - M^2 > 0$, which implies that the second fundamental form is either positive definite or negative definite (recall the Krammer's rule), any of these implies that $I\!I_{\boldsymbol{\alpha}(s)}$ never takes zero values.

16.4. Let S be a surface in $\mathbb{R}^3$ with Gauss map $\boldsymbol{N}$, and let $\boldsymbol{\beta}$ be a regular curve on S not necessarily parametrized by arc length. Show that the geodesic curvature κ_{g} of $\boldsymbol{\beta}$ is given by

$$\kappa_{\mathrm{g}} = \frac{1}{\|\boldsymbol{\beta}'\|^3} (\boldsymbol{\beta}' \times \boldsymbol{\beta}'') \cdot \boldsymbol{N}.$$

Solution:

Assume that $\boldsymbol{\beta}\colon [t_0, t_1] \longrightarrow S$ is the parametrization of the curve. Let us first parametrize the curve by arc length, i.e., set

$$s = \varphi(t) := \int_{t_0}^{t} \|\boldsymbol{\beta}'(u)\| \,\mathrm{d}u,$$

then $\mathrm{d}s/\,\mathrm{d}t = \varphi'(t) = \|\boldsymbol{\beta}'(t)\|$ and we set

$$\boldsymbol{\alpha} := \boldsymbol{\beta} \circ \varphi^{-1}, \qquad \text{i.e.} \qquad \boldsymbol{\alpha}(s) := \boldsymbol{\beta}(\varphi^{-1}(s)) = \boldsymbol{\beta}(t)$$

if $t = \varphi(s)$. Clearly (as we did in the first term),

$$\boldsymbol{\alpha}'(s) = (\varphi^{-1})'(s)\boldsymbol{\beta}'(\varphi^{-1}(s)) = \frac{1}{\|\boldsymbol{\beta}'(t)\|}\boldsymbol{\beta}'(t)$$

since $(\varphi^{-1})'(s) = 1/\varphi'(t) = 1/\|\boldsymbol{\beta}'(t)\|$ which we can also write formally as

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{1}{\|\boldsymbol{\beta}'(t)\|}\frac{\mathrm{d}}{\mathrm{d}t}.$$

Moreover,

$$\begin{aligned}
\boldsymbol{\alpha}''(s) = \frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{1}{\|\boldsymbol{\beta}'(t)\|}\boldsymbol{\beta}'(t)\Big) &= \frac{1}{\|\boldsymbol{\beta}'(t)\|}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\|\boldsymbol{\beta}'(t)\|}\boldsymbol{\beta}'(t)\Big) \\
&= \underbrace{\frac{1}{\|\boldsymbol{\beta}'(t)\|}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\|\boldsymbol{\beta}'(t)\|}\Big)\boldsymbol{\beta}'(t)}_{\text{proportional to } \boldsymbol{\alpha}'} + \frac{1}{\|\boldsymbol{\beta}'(t)\|^2}\boldsymbol{\beta}''(t).
\end{aligned}$$

Let now $\boldsymbol{N}$ be the normal to the surface (at $\boldsymbol{\alpha}(s) = \boldsymbol{\beta}(t)$). We have

$$\begin{aligned}\kappa_{\rm g}(s) = \boldsymbol{\alpha}''(s) \cdot (\boldsymbol{N}(\boldsymbol{\alpha}(s)) \times \boldsymbol{\alpha}'(s)) &= \frac{1}{\|\boldsymbol{\beta}'(t)\|^2} \boldsymbol{\beta}''(t) \cdot (\boldsymbol{N}(\boldsymbol{\alpha}(s)) \times \boldsymbol{\alpha}'(s)) \\ &= \frac{1}{\|\boldsymbol{\beta}'(t)\|^3} \boldsymbol{\beta}''(t) \cdot (\boldsymbol{N}(\boldsymbol{\beta}(t)) \times \boldsymbol{\beta}'(t)) \\ &= \frac{1}{\|\boldsymbol{\beta}'(t)\|^3} (\boldsymbol{\beta}'(t) \times \boldsymbol{\beta}''(t)) \cdot \boldsymbol{N}(\boldsymbol{\beta}(t))\end{aligned}$$

as $\boldsymbol{\beta}'$ is proportional to $\boldsymbol{\alpha}'$, hence orthogonal to $\boldsymbol{N} \times \boldsymbol{\alpha}'$ (for the second equality) and where we used

$$\boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}.$$

In particular, we have shown the desired formula.

16.5. Let S be Enneper's surface (see Problem 4.2) parametrized by

$$\boldsymbol{x}(u, v) = \Big(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\Big), \qquad (u, v) \in \mathbb{R}^2.$$

(a) Calculate the lines of curvature.

(b) Show that the asymptotic curves are given by $u \pm v = \text{const}$.

Solution:

We have calculated the coefficients of the first and second fundamental form w.r.t. $\boldsymbol{x}$ in Problem 4.2 as

$$E(u, v) = G(u, v) = (1 + u^2 + v^2)^2, \ F(u, v) = 0 \qquad \text{and} \qquad L = 2, \ M = 0, \ N = -2;$$

(a) Since the parametrization if *principal* (i.e., $F = 0$ and $M = 0$), the lines of curvature are just the coordinate curves (see, e.g., Prop. 11.18, or this can be easily computed explicitly). Hence they are given by $s \mapsto \boldsymbol{x}(s, v_0)$ and $s \mapsto \boldsymbol{x}(u_0, s)$ for u_0, $v_0 \in \mathbb{R}$.

(b) A curve $\boldsymbol{\alpha}$ with local parametrization $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ is an *asymptotic curve* if $\kappa_{\rm n} = 0$, i.e., if $I\!I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s) = 0$, or,

$$(u')^2 L + 2u'v'M + (v')2N = 0$$

Here it means that

$$2(u')^2 - 2(v')^2 = 2(u' + v')(u' - v') = 0 \qquad \text{or, equivalently} \qquad (u - v)' = 0 \text{ or } (u + v)' = 0,$$

which is equivalent to $u \pm v = \text{const}$.

16.6. (a) $(*)$ Show that the asymptotic curves on the surface given by $x^2 + y^2 - z^2 = 1$ are straight lines.

(b) Let S be a ruled surface. What are necessary and sufficient assumptions on S for all asymptotic curves being straight lines?

Hint: use linear algebra.

Solution:

(a) The surface is a one-sheeted hyperboloid, so it is doubly ruled (i.e. there are two lines through every point). As we have already proved, all lines are asymptotic curves, so we only need to prove that there are no others.

If $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a basis of T_pS consisting of eigenvectors of $-d_p\boldsymbol{N}$, then $I\!I_p(\boldsymbol{e}_i) = \kappa_i$, where κ_i are principal curvatures, and $\langle \boldsymbol{e}_1, \boldsymbol{e}_2\rangle = 0$ (there are no umbilic points since $K < 0$ everywhere). Therefore,

$$I\!I_p(a\boldsymbol{e}_1 + b\boldsymbol{e}_2) = a^2\kappa_1 + b^2\kappa_2,$$

which vanishes in the only case when $b = \pm a\sqrt{-\kappa_1/\kappa_2}$, so there are exactly two directions on which $I\!I_p$ vanishes. This completes the proof.

Equivalently, we could say that any indefinite form of rank 2 looks like $x^2 - y^2$ in some basis, so there are two vectors with zero value only.

Alternatively, one could parametrize the hyperboloid as a ruled surface via

$$\boldsymbol{x}(u, v) = \big(\cos(u), \sin(u), 0) + v(\sin(u), -\cos(u), 1),$$

then compute the coefficients of the second fundamental form, solve the differential equation

$$(u')^2L(u, v) + 2u'v'M(u, v) + (v')^2N(u, v) = 0$$

and observe that the solutions will be precisely the lines.

(b) The proof of (a) can be applied to any doubly ruled surface, so for these surfaces indeed all the asymptotic curves are lines. The statement is obviously true for planes as well. Let us prove that for all other surfaces the statement does not hold.

So, assume that S is neither a plane nor a doubly ruled surface. As we have already seen above, since S is ruled all the points are either hyperbolic or flat, which means that there are no umbilic points (except for some isolated planar ones), and every point $p \in S$ has precisely two directions on which $I\!I_p$ vanishes, one of which is the direction of the ruling. Note that these lines do not intersect each other in a ruled surface, so we can take another asymptotic curve through every point which will not be a line (formally speaking here we use the theorem of existence of a solution of differential equation with given initial data).