Durham University
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Differential Geometry 13.02.2020

**Homework 15-16**
**Starred problems due on Thursday, 27 February.**

## Christoffel symbols and Gauss' Theorema Egregium

**15.1.** Show that the Gauss curvature $K$ of the surface of revolution locally parametrized by

$$\boldsymbol{x}(u,v) = (f(v)\cos(u), f(v)\sin(u), g(v)), \qquad (u,v) \in U,$$

(for some suitable parameter domain $U$) is given by

$$K = \frac{1}{2ff'}\Big(\frac{1}{1+(f'/g')^2}\Big)'.$$

If the generating curve is parametrized by arc length, show that $K = -f''/f$. Deduce Theorema Egregium in the latter case.

**15.2.** Let $\boldsymbol{x}\colon U \longrightarrow S$ be a parametrization of a surface $S$ for which $E = G = 1$ and $F = \cos(uv)$ (so that $uv$ is the angle between the coordinate curves). Determine a suitable parameter domain $U$ on which $\boldsymbol{x}(U)$ is a surface (i.e., where the coordinate curves are not tangential). Show that

$$K = -\frac{1}{\sin(uv)}.$$

**15.3.** $(*)$ If the coefficients of the first fundamental form of a surface $S$ are given by

$$E = 2+v^2, \qquad F = 1, \qquad G = 1,$$

show that the Gauss curvature of $S$ is given by

$$K = -\frac{1}{(1+v^2)^2}.$$

**15.4.** Let $\boldsymbol{x}$ be a local parametrization of a surface $S$ such that $E = 1$, $F = 0$ and $G$ is a function of $u$ only. Show that

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{G_u}{2G}, \quad \Gamma^1_{22} = -\frac{G_u}{2}$$

and that all the other Christoffel symbols are zero. Hence show that the Gauss curvature $K$ of $S$ is given by

$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}.$$

## Curves on surfaces

**16.1.** Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be an orthonormal basis of $T_pS$ consisting of eigenvectors of the Weingarten map $-d_p\boldsymbol{N}$ with corresponding eigenvalues $\kappa_1$, $\kappa_2$. If $\boldsymbol{e} = (\cos\vartheta)\boldsymbol{e}_1 + (\sin\vartheta)\boldsymbol{e}_2$, show, that the normal curvature $\kappa_{\mathrm{n}}$ of a curve tangential to $\boldsymbol{e}$ is given by

$$\kappa_{\mathrm{n}}(\vartheta) = \kappa_1 \cos^2\vartheta + \kappa_2 \sin^2\vartheta.$$

Deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} \kappa_{\mathrm{n}}(\vartheta)\,\mathrm{d}\vartheta = H,$$

where $H$ denotes the mean curvature of $S$ at $p$. (This justifies the term *mean curvature*).

**16.2.** Let $\boldsymbol{\alpha}$ be the curve defined by

$$\boldsymbol{\alpha}(t) = \varepsilon^t(\cos t, \sin t, 1) \qquad \text{for } t \in \mathbb{R}.$$

Observe that $\boldsymbol{\alpha}$ lies on the circular cone $S = \{\,(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\,\}$.

Show that the normal curvature of $\boldsymbol{\alpha}$ in $S$ is inversely proportional to $\varepsilon^t$.

**16.3.** Show that an asymptotic curve can only exist in the hyperbolic or flat region $\{\,p \in S \mid K(p) \le 0\,\}$. (In other words, if a surface is elliptic everywhere, then there is no asymptotic curve.)

**16.4.** Let $S$ be a surface in $\mathbb{R}^3$ with Gauss map $\boldsymbol{N}$, and let $\boldsymbol{\beta}$ be a regular curve on $S$ not necessarily parametrized by arc length. Show that the geodesic curvature $\kappa_{\mathrm{g}}$ of $\boldsymbol{\beta}$ is given by

$$\kappa_{\mathrm{g}} = \frac{1}{\|\boldsymbol{\beta}'\|^3}(\boldsymbol{\beta}' \times \boldsymbol{\beta}'') \cdot \boldsymbol{N}.$$

**16.5.** Let $S$ be Enneper's surface (see Problem 4.2) parametrized by

$$\boldsymbol{x}(u,v) = \Big(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2v, u^2 - v^2\Big), \qquad (u,v) \in \mathbb{R}^2.$$

(a) Calculate the lines of curvature.

(b) Show that the asymptotic curves are given by $u \pm v = \text{const}$.

**16.6.** (a) $(*)$ Show that the asymptotic curves on the surface given by $x^2 + y^2 - z^2 = 1$ are straight lines.

(b) Let $S$ be a ruled surface. What are necessary and sufficient assumptions on $S$ for all asymptotic curves being straight lines?

*Hint:* use linear algebra.