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# Solutions 17-18

**17.1.** If $\boldsymbol{x}$ is a local parametrization of a surface $S$ in $\mathbb{R}^3$ with $E = 1$, $F = 0$ and $G$ is a function of $u$ only, write down the equations for $s \mapsto \boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ to be a geodesic. Conclude that the coordinate curves, where $v$ is constant, are geodesics.

*Solution:*

The curve $\boldsymbol{\alpha}$ is a geodesic iff

$$u''E + \frac{1}{2}(u')^2E_u + u'v'E_v + (v')^2\Big(F_v - \frac{1}{2}G_u\Big) + v''F = 0$$
$$v''G + \frac{1}{2}(v')^2G_v + u'v'G_u + (u')^2\Big(F_u - \frac{1}{2}E_v\Big) + u''F = 0,$$

which reduces here to

$$u'' - \frac{1}{2}G_u(v')^2 = 0$$
$$v''G + u'v'G_u = 0.$$

Now, for a coordinate curve with $v$ constant, we have $v' = 0$ and $v'' = 0$, so that the second equation is fulfilled. Moreover, the first one then becomes

$$u'' = 0.$$

Since the speed of $\boldsymbol{\alpha}$ is constant, we have

$$\|\boldsymbol{\alpha}'(s)\|^2 = (u')^2 + G(v')^2 = \text{const.}$$

Since $v' = 0$, we must have $u' \neq 0$ (otherwise $\boldsymbol{\alpha}'(s) = \mathbf{0}$), so that $u'' = 0$ as desired. Therefore $u(s) = u_0 + as$ (with $a \in \mathbb{R} \setminus \{0\}$) and the geodesic has the form

$$\boldsymbol{\alpha}(s) = \boldsymbol{x}(u_0 + as, v_0)$$

for some $(u_0, v_0)$ in the parameter domain and some $a \in \mathbb{R}$.

**17.2.** Let $\boldsymbol{x}\colon U \longrightarrow S$ be a parametrization of a surface $S$, and let $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ be a curve parametrized by arc length. Find an expression for the geodesic curvature $\kappa_g$ of $\boldsymbol{\alpha}$ involving $u'$, $v'$, $u''$, $v''$, $E$, $F$, $G$, $\Gamma^i_{jk}$ (i.e. the *geodesic curvature is intrinsic*, $\kappa_g$ depends only on the curve and the first fundamental form of the surface).

*Solution:*

The geodesic curvature is given by $\kappa_g = \boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \times \boldsymbol{\alpha}')$. Using the definition of the normal vector, and

$$\boldsymbol{\alpha}' = u'\boldsymbol{x}_u + v'\boldsymbol{x}_v$$

and its derivative

$$\boldsymbol{\alpha}'' = u''\boldsymbol{x}_u + (u')^2\boldsymbol{x}_{uu} + 2u'v'\boldsymbol{x}_{uv} + (v')^2\boldsymbol{x}_{vv} + v''\boldsymbol{x}_v$$

we obtain

$$
\begin{aligned}
\kappa_{\mathrm{g}} &= \boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \times \boldsymbol{\alpha}') \\
&= \frac{1}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|} \big(u''\boldsymbol{x}_u + (u')^2\boldsymbol{x}_{uu} + 2u'v'\boldsymbol{x}_{uv} + (v')^2\boldsymbol{x}_{vv} + v''\boldsymbol{x}_v\big) \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times (u'\boldsymbol{x}_u + v'\boldsymbol{x}_v)\big) \\
&= \frac{1}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|} \Big(v'u''\boldsymbol{x}_u \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) \\
&\qquad + (u')^3\boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) + (u')^2v'\boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) \\
&\qquad + 2(u')^2v'\boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) + 2u'(v')^2\boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) \\
&\qquad + u'(v')^2\boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) + (v')^3\boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) \\
&\qquad + u'v''\boldsymbol{x}_v \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big)\Big).
\end{aligned}
$$

Note first that

$$\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|^2 = EG - F^2$$

We now have to understand the expressions $\boldsymbol{x}_u \cdot ((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v)$ etc. Using the rule

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \quad \text{or, equivalently,} \quad (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$$

we obtain

$$
\begin{aligned}
\boldsymbol{x}_u \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) &= \boldsymbol{x}_u \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_v)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_v)\boldsymbol{x}_u\big) = F^2 - EG \\
\boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) &= \boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_u)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_u)\boldsymbol{x}_u\big) = E\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v - F\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u \\
&= E\Big(F_u - \frac{1}{2}E_v\Big) - \frac{1}{2}FE_u \\
&= (EG - F^2)\Gamma^2_{11} \\
\boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) &= \boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_v)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_v)\boldsymbol{x}_u\big) = F\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v - G\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u \\
&= F\Big(F_u - \frac{1}{2}E_v\Big) - \frac{1}{2}GE_u \\
&= -(EG - F^2)\Gamma^1_{11} \\
\boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) &= \boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_u)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_u)\boldsymbol{x}_u\big) = E\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v - F\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u \\
&= \frac{1}{2}EG_u - \frac{1}{2}E_vF \\
&= (EG - F^2)\Gamma^2_{12} \\
\boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) &= \boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_v)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_v)\boldsymbol{x}_u\big) = F\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v - G\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u \\
&= \frac{1}{2}FG_u - \frac{1}{2}E_vG \\
&= -(EG - F^2)\Gamma^1_{12} \\
\boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) &= \boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_u)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_u)\boldsymbol{x}_u\big) = E\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v - F\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u \\
&= \frac{1}{2}EG_v - F\Big(F_v - \frac{1}{2}G_u\Big) \\
&= (EG - F^2)\Gamma^2_{22} \\
\boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) &= \boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_v)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_v)\boldsymbol{x}_u\big) = F\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v - G\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u \\
&= \frac{1}{2}FG_v - G\Big(F_v - \frac{1}{2}G_u\Big) \\
&= -(EG - F^2)\Gamma^1_{22} \\
\boldsymbol{x}_v \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) &= \boldsymbol{x}_v \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_u)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_u)\boldsymbol{x}_u\big) = EG - F^2.
\end{aligned}
$$

Alltogether, we have

$$\begin{aligned}
\kappa_{\mathrm{g}} = \frac{1}{\sqrt{EG-F^2}}\Big((u'v''-u''v')(EG-F^2) \\
&+ (u')^3\Big(E\Big(F_u-\frac{1}{2}E_v\Big)-\frac{1}{2}FE_u\Big)+(u')^2v'\Big(F\Big(F_u-\frac{1}{2}E_v\Big)-\frac{1}{2}GE_u\Big) \\
&+ 2(u')^2v'\Big(\frac{1}{2}EG_u-\frac{1}{2}E_vF\Big)+2u'(v')^2\Big(\frac{1}{2}FG_u-\frac{1}{2}E_vG\Big) \\
&+ u'(v')^2\Big(\frac{1}{2}EG_v-F\Big(F_v-\frac{1}{2}G_u\Big)\Big)+(v')^3\Big(\frac{1}{2}FG_v-G\Big(F_v-\frac{1}{2}G_u\Big)\Big)\Big) \\
&= \sqrt{EG-F^2}\big(\Gamma_{11}^2u'^3-\Gamma_{22}^1v'^3+(2\Gamma_{12}^2-\Gamma_{11}^1)u'^2v'-(2\Gamma_{12}^1-\Gamma_{22}^2)u'v'^2-u''v'+v''u'\big)
\end{aligned}$$

In particular, for an arbitrarily parametrized curve the geodesic curvatire can be computed as

$$\kappa_{\mathrm{g}} = \frac{\sqrt{EG-F^2}\big(\Gamma_{11}^2u'^3-\Gamma_{22}^1v'^3+(2\Gamma_{12}^2-\Gamma_{11}^1)u'^2v'-(2\Gamma_{12}^1-\Gamma_{22}^2)u'v'^2-u''v'+v''u'\big)}{(Eu'^2+2Fu'v'+Gv'^2)^{3/2}}$$

(cf. Exercise 6.4).

**17.3.** Show that a curve of constant geodesic curvature $c$ on the unit sphere $S^2(1)$ in $\mathbb{R}^3$ is a planar circle of length $2\pi(1+c^2)^{-1/2}$.

*Hint:* If $\boldsymbol{\alpha}$ is a curve of constant geodesic curvature $c$ show that the vector $\boldsymbol{e}(s)=\boldsymbol{\alpha}(s)\times\boldsymbol{\alpha}'(s)+c\boldsymbol{\alpha}(s)$ does not depend on $s$, where $(\cdot)'$ denotes differentiation with respect to arc length).

*Solution:*

On the unit sphere we have $\boldsymbol{N}(\boldsymbol{\alpha}(s))=\boldsymbol{\alpha}(s)$. Therefore,

$$\begin{aligned}
\boldsymbol{e}(s) &= \boldsymbol{\alpha}(s)\times\boldsymbol{\alpha}'(s)+c\boldsymbol{\alpha}(s), \\
\boldsymbol{e}'(s) &= \underbrace{\boldsymbol{\alpha}'(s)\times\boldsymbol{\alpha}'(s)}_{=\boldsymbol{0}}+\underbrace{\boldsymbol{\alpha}(s)}_{=\boldsymbol{N}(\boldsymbol{\alpha}(s))}\times\boldsymbol{\alpha}''(s)+c\boldsymbol{\alpha}'(s), \\
&= \boldsymbol{N}(\boldsymbol{\alpha}(s))\times\Big(\kappa_{\mathrm{n}}\boldsymbol{N}(\boldsymbol{\alpha}(s))+\underbrace{\kappa_{\mathrm{g}}}_{=c}\big(\boldsymbol{N}(\boldsymbol{\alpha}(s))\times\boldsymbol{\alpha}'(s)\big)\Big)+c\boldsymbol{\alpha}'(s), \\
&= c\boldsymbol{N}(\boldsymbol{\alpha}(s))\times\big(\boldsymbol{N}(\boldsymbol{\alpha}(s))\times\boldsymbol{\alpha}'(s)\big)+c\boldsymbol{\alpha}'(s).
\end{aligned}$$

Now, note that $\boldsymbol{a}:=\boldsymbol{\alpha}'(s)$ and $\boldsymbol{b}:=\boldsymbol{N}(\boldsymbol{\alpha}(s))$ are orthonormal vectors, therefore $\boldsymbol{c}:=\boldsymbol{a}\times\boldsymbol{b}$ is also a unit vector orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$. In particular, $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$ is a positively oriented orthonormal basis of $\mathbb{R}^3$. For such a basis, we have $\boldsymbol{b}\times(\boldsymbol{b}\times\boldsymbol{a})=-\boldsymbol{a}$, and hence $\boldsymbol{e}'(s)=\boldsymbol{0}$, so $\boldsymbol{e}(s)=\boldsymbol{e}$ is a constant vector.

We will now show that $\boldsymbol{\alpha}(s)$ lies in a plane: We have

$$\boldsymbol{\alpha}(s)\cdot\boldsymbol{e}=\boldsymbol{\alpha}(s)\cdot\big(\boldsymbol{\alpha}(s)\times\boldsymbol{\alpha}'(s)+c\boldsymbol{\alpha}(s)\big)=c\boldsymbol{\alpha}(s)\cdot\boldsymbol{\alpha}(s)=c$$

for all $s\in\mathbb{R}$ as $\boldsymbol{\alpha}(s)\in S^2(1)$. But this means that $\boldsymbol{\alpha}(s)$ makes a constant angle with $\boldsymbol{e}$ and thus lies in a plane at distance $c/\|\boldsymbol{e}(s)\|$ from the origin. Since $\|\boldsymbol{e}(s)\|=\sqrt{1+c^2}$ (by Pythagoras' theorem: $\{\boldsymbol{\alpha}(s)\times\boldsymbol{\alpha}'(s),\boldsymbol{\alpha}'(s)\}$ are orthonormal), the radius of the circle (the intersection of the plane with the unit sphere) is $r=\sqrt{1-c^2/(1+c^2)}=1/\sqrt{1+c^2}$. Hence, the circle has circumference $2\pi r=2\pi/\sqrt{1+c^2}$.

**17.4.** $(*)$ Let $S$ be a surface in $\mathbb{R}^3$ and suppose that $\Pi$ is a plane which intersects $S$ orthogonally along a regular curve $\boldsymbol{\gamma}$. If $\boldsymbol{\alpha}(s)$ is a parametrization of $\boldsymbol{\gamma}$ such that $\|\boldsymbol{\alpha}'(s)\|$ is constant, show that $\boldsymbol{\alpha}$ is a geodesic of $S$.

*Solution:*

By construction, we have that $\boldsymbol{\alpha}'(s)$ and the normal $\boldsymbol{N}(\boldsymbol{\alpha}(s))$ are parallel to $\Pi$ for all $s$. Let $\boldsymbol{e}$ be a non-zero vector normal to $\Pi$, then $\boldsymbol{N}(\boldsymbol{\alpha}(s)) \times \boldsymbol{\alpha}'(s)$ is parallel to $\boldsymbol{e}$. From $\boldsymbol{\alpha}'(s) \cdot \boldsymbol{e} = 0$ (again, $\boldsymbol{\alpha}'(s)$ is parallel to $\Pi$) we deduce that (after taking the derivative) $\boldsymbol{\alpha}''(s) \cdot \boldsymbol{e} = 0$ (as $\boldsymbol{e}$ is independent of $s$), so we see that

$$\kappa_{\mathrm{g}} = \frac{1}{\|\boldsymbol{\alpha}'(s)\|^3}\big(\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)\big) \cdot \boldsymbol{N}(\boldsymbol{\alpha}(s)) = \frac{1}{\|\boldsymbol{\alpha}'(s)\|^3}\big(\boldsymbol{N}(\boldsymbol{\alpha}(s)) \times \boldsymbol{\alpha}'(s)\big) \cdot \boldsymbol{\alpha}''(s) = 0.$$

Therefore, since the curve is also parametrized proportionally to arc length, it is a geodesic.

**17.5.** (a) Show that any constant speed curve on a surface $S$ in $\mathbb{R}^3$ which is a curve of intersection of $S$ with a plane of reflectional symmetry of $S$ is a geodesic.

(b) Show that the curves of intersection of the coordinate planes in $\mathbb{R}^3$ with the surface $S$ defined by the equation $x^4 + y^6 + z^8 = 1$ are geodesics.

*Solution:*

(a) A plane of reflection leaving a surface invariant intersects the surface orthogonally (prove this!). Therefore, the result follows immediately from the previous exercise.

(b) Note that the reflections $(x, y, z) \mapsto (-x, y, z)$, $(x, y, z) \mapsto (x, -y, z)$ and $(x, y, z) \mapsto (x, -y, z)$ all leave the surface given by $x^4 + y^6 + z^8 = 1$ invariant. Since these reflections are reflections along the coordinate planes, the result follows.

**17.6.** Let $\boldsymbol{\alpha}$ be a regular curve on a surface $S$ in $\mathbb{R}^3$.

(a) If $\boldsymbol{\alpha}$ is both a line of curvature and a geodesic, show that $\boldsymbol{\alpha}$ is a planar curve.
*Hint:* Show that $\boldsymbol{N} \times \boldsymbol{\alpha}'$ is constant along $\boldsymbol{\alpha}$.

(b) If $\boldsymbol{\alpha}$ is both a geodesic and a planar curve with nowhere vanishing curvature show that $\boldsymbol{\alpha}$ is a line of curvature.

*Solution:*

(a) Denote

$$\boldsymbol{e}(s) := (\boldsymbol{N} \circ \boldsymbol{\alpha})(s) \times \boldsymbol{\alpha}'(s)$$

so that its derivative is

$$\begin{aligned}\boldsymbol{e}'(s) &= (\boldsymbol{N} \circ \boldsymbol{\alpha})'(s) \times \boldsymbol{\alpha}'(s) + (\boldsymbol{N} \circ \boldsymbol{\alpha})(s) \times \boldsymbol{\alpha}''(s) \\ &= (d_{\boldsymbol{\alpha}(s)}\boldsymbol{N})(\boldsymbol{\alpha}'(s)) \times \boldsymbol{\alpha}'(s) + \boldsymbol{N}(\boldsymbol{\alpha}(s)) \times \boldsymbol{\alpha}''(s)\end{aligned}$$

Now, since $\boldsymbol{\alpha}$ is a line of curvature, $d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s))$ is a multiple of $\boldsymbol{\alpha}'(s)$, hence the first vector product vanishes (as $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$), and for the second term, note that as $\boldsymbol{\alpha}$ is a geodesic, $\boldsymbol{\alpha}''(s)$ is a multiple of $\boldsymbol{N}(\boldsymbol{\alpha}(s))$, and hence this vector product also vanishes. Alltogether we have shown $\boldsymbol{e}'(s) = 0$ for all $s$, say

$$\boldsymbol{e}(s) = \boldsymbol{e}_0$$

for some vector $\boldsymbol{e}_0 \in \mathbb{R}^3$. Note that $\boldsymbol{e}_0 \neq \boldsymbol{0}$, because $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}''$ are orthogonal (here we use the constant speed property of a geodesic), so $\boldsymbol{\alpha}'$ and $\boldsymbol{N}(\boldsymbol{\alpha}(s))$ are also orthogonal (and of course non-zero).

Let us now show that $\boldsymbol{\alpha}(s)$ lies in a plane normal to $\boldsymbol{e}_0$, i.e., $\boldsymbol{\alpha}(s) \cdot \boldsymbol{e}_0 = \text{const}$, or equivalently, $\boldsymbol{\alpha}'(s) \cdot \boldsymbol{e}_0 = 0$. Indeed,

$$\boldsymbol{\alpha}' \cdot \boldsymbol{e}_0 = \boldsymbol{\alpha}' \cdot \big((\boldsymbol{N} \circ \boldsymbol{\alpha}) \times \boldsymbol{\alpha}'\big) = (\boldsymbol{N} \circ \boldsymbol{\alpha}) \cdot \big(\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'\big) = 0.$$

In particular, $\boldsymbol{\alpha}(s)$ lies in a plane for all $s$.

(b) If $\boldsymbol{\alpha}$ is a geodesic, then $\boldsymbol{\alpha}'' = \kappa_{\mathrm{n}} \boldsymbol{N}$. Moreover, there exist $\boldsymbol{e}_0 \in \mathbb{R}^3$ such that $\boldsymbol{\alpha}(s) \cdot \boldsymbol{e}_0 = \text{const}$ for all $s$ (as $\boldsymbol{\alpha}$ lies in a plane), hence taking the derivatives give $\boldsymbol{\alpha}' \cdot \boldsymbol{e}_0 = 0$ and $\boldsymbol{\alpha}'' \cdot \boldsymbol{e}_0 = 0$. Using the fact that $\kappa_{\mathrm{n}}(s) \neq 0$ for all $s$ we conclude that $\boldsymbol{e}_0$ is orthogonal to $\boldsymbol{N}(\boldsymbol{\alpha}(s))$ and $\boldsymbol{\alpha}'(s)$ for all $s$. Taking the derivative of $\boldsymbol{N}(\boldsymbol{\alpha}(s)) \cdot \boldsymbol{e}_0 = 0$ gives

$$d_{\boldsymbol{\alpha}(s)} \boldsymbol{N}(\boldsymbol{\alpha}'(s)) \cdot \boldsymbol{e}_0 = 0$$

for all $s$, and from the fact that $\boldsymbol{N}$ is a unit vector, we also obtain that $d_{\boldsymbol{\alpha}(s)} \boldsymbol{N}(\boldsymbol{\alpha}'(s))$ is orthognal to $\boldsymbol{N}(\boldsymbol{\alpha}(s))$. In particular we have shown that $d_{\boldsymbol{\alpha}(s)} \boldsymbol{N}(\boldsymbol{\alpha}'(s))$ and $\boldsymbol{\alpha}'(s)$ both are orthogonal to $\boldsymbol{e}_0$ and $d_{\boldsymbol{\alpha}(s)} \boldsymbol{N}(\boldsymbol{\alpha}'(s))$, hence there must be a scalar $\lambda(s) \in \mathbb{R}$ such that $d_{\boldsymbol{\alpha}(s)} \boldsymbol{N}(\boldsymbol{\alpha}'(s)) = \lambda(s) \boldsymbol{\alpha}'(s)$, i.e., $\boldsymbol{\alpha}$ is a line of curvature.

**18.1.** Find all the geodesics on the flat torus $S^1(1) \times S^1(1) \subset \mathbb{R}^4$, where $S^1(1)$ is the circle of radius 1 in $\mathbb{R}^2$ centered at the origin. Prove that there are infinitely many both closed and non-closed geodesics through the point $(1, 0, 1, 0) \in S^1(1) \times S^1(1)$.

*Solution:*

The plane $\mathbb{R}^2$ and the flat torus $T = S^1(1) \times S^1(1)$ are locally isometric via

$$f(u, v) = (\cos u, \sin u, \cos v, \sin v)$$

(as it can be easily seen from $f_u \cdot f_u = 1$, $f_u \cdot f_v = 0$ and $f_v \cdot f_v = 0$, and the fact that $E = G = 1$, $F = 0$ are also the coefficients of the first fundamental form of the plane). Local isometries preserve geodesics, hence images of lines under $f$ are geodesics of $T$: examples through $(1, 0, 1, 0)$ are

$$\boldsymbol{\alpha}_r \colon \mathbb{R} \longrightarrow T, \qquad \boldsymbol{\alpha}_{p,q}(s) = (\cos(ps), \sin(ps), \cos(qs), \sin(qs)).$$

for some $p, q \in \mathbb{R}$ such that $p^2 + q^2 = 1$ (these are images of the lines $s \mapsto (ps, qs)$ in the plane, having unit speed). Note that $\boldsymbol{\alpha}_r(0) = (1, 0, 1, 0)$. Moreover, if $r = p/q$ is rational (w.l.o.g., $p, q$ both rational, say, $p = a/c$ and $q = b/c$, $a, b \in \mathbb{Z}$, $c \in \mathbb{N}$), then $\boldsymbol{\alpha}_{p,q}(s + 2\pi c) = \boldsymbol{\alpha}_{p,q}(s)$ and hence $\boldsymbol{\alpha}_{p,q}$ is a closed geodesic on $T$. Obviously, there are infinitely many such parameters $p$ and $q$.

If $p/q$ is irrational, then $\boldsymbol{\alpha}_{p,q}(s_1) = \boldsymbol{\alpha}_{p,q}(s_2)$ implies $p(s_1 - s_2), q(s_1 - s_2) \in 2\pi\mathbb{Z}$, i.e.,

$$2\pi(s_1 - s_2) \in (p^{-1}\mathbb{Z}) \cap (q^{-1}\mathbb{Z}).$$

But since $p/q$ is irrational, the latter set only contains $\{0\}$, and hence $s_1 = s_2$, i.e., the curve $\boldsymbol{\alpha}_{p,q}$ is injective, i.e., the line $\mathbb{R}$ is embedded injectively into $T$. Again, there are infinitely many such parameters $p$ and $q$.

**18.2.** Let $\mathbb{H}$ be the hyperbolic plane, i.e. the surface $\mathbb{R} \times (0, \infty)$ with coefficients of the first fundamental form $E(u, v) = G(u, v) = 1/v^2$ and $F(u, v) = 0$. Show that the geodesics in $\mathbb{H}$ are the intersections of $\mathbb{H}$ with the lines and circles in $\mathbb{R}^2$ which meet the $u$-axis orthogonally.

*Hint:* After obtaining the differential equations you may not try to solve them but, instead, just check that the curves above are indeed geodesics, and then prove that there are no others.

*Solution:*

The equation of a geodesic in a local parametrization is

$$\begin{aligned} u''E + \frac{1}{2}(u')^2 E_u + u'v'E_v + (v')^2\Big(F_v - \frac{1}{2}G_u\Big) + v''F = 0, \\ v''G + \frac{1}{2}(v')^2 G_v + u'v'G_u + (u')^2\Big(F_u - \frac{1}{2}E_v\Big) + u''F = 0, \end{aligned}$$

which reduces here to

$$\begin{aligned} \frac{1}{v^2}u'' - 2\frac{1}{v^3}u'v' = 0, \\ \frac{1}{v^2}v'' - \frac{1}{v^3}(v')^2 + \frac{1}{v^3}(u')^2 = 0, \end{aligned}$$

which is equivalent to

$$\Big(\frac{u'}{v^2}\Big)' = 0, \quad \text{and} \quad v'' + \frac{u'^2 - v'^2}{v} = 0.$$

These are equivalent to

$$u' = cv^2 \quad \text{and} \quad v'' + \frac{u'^2 - v'^2}{v} = 0$$

for some constant $c \in \mathbb{R}$.

Consider vertical lines first, i.e. $u = u_0$. Then the first equation clearly holds for $c = 0$, and the second reduces to $v'' = \frac{v'^2}{v}$, which also holds if we parametrize a vertical line by $v(s) = ke^s$ for any positive $k$.

Consider now semicircles orthogonal to the real axis, each of these can be parametrized by

$$\boldsymbol{\alpha}(s) = (u(s), v(s)) = (u_0 + a\cos f(s), a\sin f(s))$$

for some $u_0 \in \mathbb{R}$, $a \in \mathbb{R}_{>0}$ and a smooth monotone function $f$. The first equation then becomes

$$f'(s) = -ca\sin f(s),$$

so assume the function $f$ satisfies this. We need to verify that the second equation is then also fulfilled. In view of the relation above, we have

$$\begin{aligned} v'(s) =& af'(s)\cos f(s) = && -ca^2\sin f(s)\cos f(s) = \frac{-ca^2}{2}\sin 2f(s), \\ v''(s) =& \frac{-ca^2}{2}2f'(s)\cos 2f(s) = && c^2a^3\sin f(s)\cos 2f(s), \\ u'(s) =& ca^2\sin^2 f(s). \end{aligned}$$

Therefore,

$$v'' + \frac{u'^2 - v'^2}{v} = c^2a^3\sin f(s)\cos 2f(s) + \frac{c^2a^4\sin^4 f(s) - c^2a^4\sin^2 f(s)\cos^2 f(s)}{a\sin f(s)} = c^2a^3\sin f(s)\big(\cos 2f(s) + (\sin^2 f(s) - \cos^2 f($$

so the second equation also holds.

Finally, for a given point $p \in \mathbb{H}$ and a tangent vector $w \in T_p\mathbb{H}$ there exists a unique circle (or line) through $p$ and tangent to $w$ intersecting the real axis orthogonally. By the uniqueness theorem, this implies that there are no other geodesics except for the ones described above.

**18.3.** How many closed geodesics are there on the surface of revolution in $\mathbb{R}^3$ obtained by rotating the curve $z = 1/x^2$, $(x > 0)$ around the $z$-axis?

*Solution:*

Assume that $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ is a closed geodesic, where

$$\boldsymbol{x}(u, v) = (f(v) \cos u, f(v) \sin u, \frac{1}{f^2(v)}),$$

where $f$ is monotonic and the curve $(f(v), 1/f^2(v))$ has unit speed. The Clairaut relation then says that

$$f(v(s)) \cos \Theta(s) = \text{const},$$

where $\Theta(s)$ is the angle formed by $\boldsymbol{\alpha}'(s)$ and the parallel through $\boldsymbol{\alpha}(s)$. Let $z_{\text{min}}$ and $z_{\text{max}}$ be the minimal and maximal values of $z$ along $\boldsymbol{\alpha}$, denote $z_{\text{min}} = 1/f(v^2(s_{\text{min}}))$ and $z_{\text{max}} = 1/f^2(v(s_{\text{max}}))$, these defines the values of $s$ uniquely since $f$ is monotonic. Then $\Theta(s_{\text{min}}) = \Theta(s_{\text{max}}) = 0$, which implies that $f(v(s_{\text{min}})) = f(v(s_{\text{max}}))$, so $z_{\text{min}} = z_{\text{max}}$, i.e. $\boldsymbol{\alpha}$ must be a parallel. However, it is easy to see that no parallel is a geodesic (as $f'(v)$ never vanishes). This proves that the surface has no closed geodesics.

**18.4.** $(*)$ Let $S$ be the cone obtained by rotating the line $z = \beta x$ $(z > 0)$ around the $z$-axis, where $\beta$ is a positive constant. Let $\boldsymbol{\alpha}(s) = (x(s), y(s), z(s))$ be a geodesic on $S$ intersecting the parallel $z = 1$ at an angle $\vartheta_0$. Find the smallest value of $z(s)$. Investigate whether $\boldsymbol{\alpha}$ has self-intersections.

*Solution:*

Parametrize the generating curve by $(v, 0, \beta v)$. Then the Clairaut equation reduces to

$$v(s) \cos \vartheta(s) = \text{const},$$

where $\vartheta(s)$ is the angle formed by $\boldsymbol{\alpha}$ with the parallel at $\boldsymbol{\alpha}(s)$.The constant here is the value of $v(s) \cos \vartheta$ at $z = 1$, i.e. at $v = 1/\beta$. Thus, we have an equation

$$v(s) \cos \vartheta(s) = \frac{\cos \vartheta_0}{\beta}.$$

By symmetry, at the point $\boldsymbol{\alpha}(s_0)$ of $\boldsymbol{\alpha}$ closest to the origin the angle $\vartheta(s_0)$ is equal to zero, so $v(s_0) = \cos \vartheta_0/\beta$. Therefore,

$$z(s_0) = \beta v(s_0) = \cos(\vartheta_0),$$

so it is independent of $\beta$! Note that if $\vartheta_0 = \pi/2$, then the distance is 0 which means that the geodesic goes through the apex.

Alternatively, we could use the fact the geodesics on a cone are just images of lines under the local isometry between the plane and a cone. In particular, by considering the preimage of the cone under an isometry in $\mathbb{R}^2$, one can easily see that the $\boldsymbol{\alpha}$ is self-intersecting if and only if the total angle of the cone in the apex is strictly less that $\pi$. By Pythagoras' Theorem, the latter is equivalent to $2/\sqrt{1 + \beta^2} < 1$, which is the same as $\beta > \sqrt{3}$ or $\arctan \beta > \pi/3$.

**18.5.** Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a curve parametrized by arc length with everywhere non-zero curvature, and let $\boldsymbol{b}(s)$ be a vector such that the map

$$\boldsymbol{x}(s, v) = \boldsymbol{\alpha}(s) + v\boldsymbol{b}(s), \qquad s \in I, v \in (-\epsilon, \epsilon),$$

is a parametrization of a regular surface $S$ for some $\varepsilon > 0$ ($S$ is a ruled surface — you don't have to show that the surface is regular).

(a) Is the curve $\boldsymbol{\beta}\colon (-\varepsilon, \varepsilon) \longrightarrow S$ given by $\boldsymbol{\beta}(v) = \boldsymbol{x}(s_0, v)$ for some $s_0 \in I$ a geodesic? Justify your answer.

(b) Assume now that $\boldsymbol{b}(s)$ is the binormal of the space curve $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(s)$. Prove that $\boldsymbol{\alpha}$ is a geodesic on $S$ (i.e., show that the *generating* curve is a geodesic on the ruled surface generated by a curve and its binormal.)

*Solution:*

(a) Any line in a surface is a geodesic (as in its standard parametrisation, $\boldsymbol{\alpha}(s) = p + s\boldsymbol{v}$ has derivatives $\boldsymbol{\alpha}'(s) = \boldsymbol{v}$ and $\boldsymbol{\alpha}''(s) = \boldsymbol{0}$, hence $\kappa_{\mathrm{g}} = 0$.

(b) The normal $\boldsymbol{N}_{\boldsymbol{\alpha}}$ and binormal $\boldsymbol{b}$ of the curve $\boldsymbol{\alpha}$ are given by

$$\boldsymbol{N}_{\boldsymbol{\alpha}}(s) = \frac{1}{\|\boldsymbol{\alpha}''(s)\|}\boldsymbol{\alpha}''(s) \qquad \text{and} \qquad \boldsymbol{b}(s) = \boldsymbol{\alpha}'(s) \times \boldsymbol{N}_{\boldsymbol{\alpha}}(s)$$

(assuming that $\boldsymbol{\alpha}''(s) \neq 0$, see Section 4 of the notes of the first term). The two tangent vectors of the ruled surface are $\boldsymbol{x}_s = \boldsymbol{\alpha}'$ and $\boldsymbol{x}_v = \boldsymbol{b}$, hence the normal vector $\boldsymbol{N}$ of the surface is

$$\boldsymbol{N} = \frac{1}{\|\boldsymbol{\alpha}' \times \boldsymbol{b}\|}\boldsymbol{\alpha}' \times \boldsymbol{b},$$

and hence, the vector $\boldsymbol{N} \times \boldsymbol{\alpha}'$ is proportional to $(\boldsymbol{\alpha}' \times \boldsymbol{b}) \times \boldsymbol{\alpha}'$ and therefore proportional to $\boldsymbol{b}$ (since $\boldsymbol{\alpha}'$ and $\boldsymbol{b}$ are orthonormal). Now, $\boldsymbol{b}$ is, by definition, orthogonal to $\boldsymbol{\alpha}''$ and hence $\kappa_{\mathrm{g}} = \boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \times \boldsymbol{\alpha}') = 0$.

**Alternative solution:** You can also verify that $\boldsymbol{\alpha}''(s)$ is orthogonal to $T_{\boldsymbol{\alpha}(s)}S$ for all $s$ by checking

$$\boldsymbol{\alpha}'' \cdot \boldsymbol{x}_s = \boldsymbol{\alpha}'' \cdot \boldsymbol{\alpha}' \overset{!}{=} 0 \qquad \text{and} \qquad \boldsymbol{\alpha}'' \cdot \boldsymbol{x}_v = \boldsymbol{\alpha}'' \cdot \boldsymbol{b} \overset{!}{=} 0.$$

Now, $\boldsymbol{\alpha}'' \cdot \boldsymbol{\alpha}' = 0$ as $\|\boldsymbol{\alpha}'\|^2 = 1$, which implies $2\boldsymbol{\alpha}'' \cdot \boldsymbol{\alpha}' = 0$. Moreover, $\boldsymbol{b}$ is by definition orthogonal to $\boldsymbol{\alpha}''$, and hence the second orthogonality condition is also satisfied.