Durham University
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Differential Geometry 27.02.2020

Homework 17-18
Starred problems due on Tuesday, 10 March.

Curves on surfaces. Geodesics.

17.1. If $\boldsymbol{x}$ is a local parametrization of a surface S in $\mathbb{R}^3$ with $E = 1$, $F = 0$ and G is a function of u only, write down the equations for $s \mapsto \boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ to be a geodesic. Conclude that the coordinate curves, where v is constant, are geodesics.

17.2. Let $\boldsymbol{x}: U \longrightarrow S$ be a parametrisation of a surface S, and let $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ be a curve parametrised by arclength. Find an expression for the geodesic curvature κ_{g} of $\boldsymbol{\alpha}$ involving u', v', u'', v'', E, F, G, Γ^i_{jk} (i.e. the *geodesic curvature is intrinsic*, κ_{g} depends only on the curve and the first fundamental form of the surface).

17.3. Show that a curve of constant geodesic curvature c on the unit sphere $S^2(1)$ in $\mathbb{R}^3$ is a planar circle of length $2\pi(1 + c^2)^{-1/2}$.

Hint: If $\boldsymbol{\alpha}$ is a curve of constant geodesic curvature c show that the vector $\boldsymbol{e}(s) = \boldsymbol{\alpha}(s) \times \boldsymbol{\alpha}'(s) + c\boldsymbol{\alpha}(s)$ does not depend on s, where $(\cdot)'$ denotes differentiation with respect to arc length).

17.4. $(*)$ Let S be a surface in $\mathbb{R}^3$ and suppose that Π is a plane which intersects S orthogonally along a regular curve $\boldsymbol{\gamma}$. If $\boldsymbol{\alpha}(s)$ is a parametrization of $\boldsymbol{\gamma}$ such that $\|\boldsymbol{\alpha}'(t)\|$ is constant, show that $\boldsymbol{\alpha}$ is a geodesic of S.

17.5. (a) Show that any constant speed curve on a surface S in $\mathbb{R}^3$ which is a curve of intersection of S with a plane of reflectional symmetry of S is a geodesic.

(b) Show that the curves of intersection of the coordinate planes in $\mathbb{R}^3$ with the surface S defined by the equation $x^4 + y^6 + z^8 = 1$ are geodesics.

17.6. Let $\boldsymbol{\alpha}$ be a regular curve on a surface S in $\mathbb{R}^3$.

(a) If $\boldsymbol{\alpha}$ is both a line of curvature and a geodesic, show that $\boldsymbol{\alpha}$ is a planar curve.
Hint: Show that $\boldsymbol{N} \times \boldsymbol{\alpha}'$ is constant along $\boldsymbol{\alpha}$).

(b) If $\boldsymbol{\alpha}$ is both a geodesic and a planar curve with nowhere vanishing curvature show that $\boldsymbol{\alpha}$ is a line of curvature.

Geodesics – 2.

18.1. Find all the geodesics on the flat torus $S^1(1) \times S^1(1) \subset \mathbb{R}^4$, where $S^1(1)$ is the circle of radius 1 in $\mathbb{R}^2$ centered at the origin. Prove that there are infinitely many both closed and non-closed geodesics through the point $(1, 0, 1, 0) \in S^1(1) \times S^1(1)$.

18.2. Let $\mathbb{H}$ be the hyperbolic plane, i.e. the surface $\mathbb{R} \times (0, \infty)$ with coefficients of the first fundamental form $E(u, v) = G(u, v) = 1/v^2$ and $F(u, v) = 0$. Show that the geodesics in $\mathbb{H}$ are the intersections of $\mathbb{H}$ with the lines and circles in $\mathbb{R}^2$ which meet the u-axis orthogonally.

Hint: After obtaining the differential equations you may not try to solve them but, instead, just check that the curves above are indeed geodesics, and then prove that there are no others.

18.3. How many closed geodesics are there on the surface of revolution in $\mathbb{R}^3$ obtained by rotating the curve $z = 1/x^2$, $(x > 0)$ around the z-axis?

18.4. $(*)$ Let S be the cone obtained by rotating the line $z = \beta x$ $(z > 0)$ around the z-axis, where β is a positive constant. Let $\boldsymbol{\alpha}(s) = (x(s), y(s), z(s))$ be a geodesic on S intersecting the parallel $z = 1$ at an angle ϑ_0. Find the smallest value of $z(s)$. Investigate whether $\boldsymbol{\alpha}$ has self-intersections.

18.5. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a curve parametrized by arc length with everywhere non-zero curvature, and let $\boldsymbol{b}(s)$ be a vector such that the map

$$\boldsymbol{x}(s, v) = \boldsymbol{\alpha}(s) + v\boldsymbol{b}(s), \qquad s \in I, v \in (-\epsilon, \epsilon),$$

is a parametrization of a regular surface S for some $\varepsilon > 0$ (S is a ruled surface — you don't have to show that the surface is regular).

(a) Is the curve $\boldsymbol{\beta}\colon (-\varepsilon, \varepsilon) \longrightarrow S$ given by $\boldsymbol{\beta}(v) = \boldsymbol{x}(s_0, v)$ for some $s_0 \in I$ a geodesic? Justify your answer.

(b) Assume now that $\boldsymbol{b}(s)$ is the binormal of the space curve $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(s)$. Prove that $\boldsymbol{\alpha}$ is a geodesic on S (i.e., show that the *generating* curve is a geodesic on the ruled surface generated by a curve and its binormal.)