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Solutions 3-4

3.1. Let $\boldsymbol{\alpha}$ denote the catenary from Exercise 2.1. Show that

(a) the involute of $\boldsymbol{\alpha}$ starting from $(0,1)$ is the tractrix from Exercise 1.6 (with x- and y-axes exchanged and different parametrization);

(b) the evolute of $\boldsymbol{\alpha}$ is the curve given by

$$\boldsymbol{\beta}(u) = (u - \sinh u \cosh u, 2\cosh u)$$

(c) Find the singular points of $\boldsymbol{\beta}$ and give a sketch of its trace.

Solution:

(a) The involute of $\boldsymbol{\alpha}$ has parametrization

$$\boldsymbol{\gamma}(u) = \boldsymbol{a}(u) - \ell(u)\boldsymbol{t}(u)$$

Since

$$\boldsymbol{a}'(u) = (1, \sinh u),$$

we have

$$\ell(u) = \int_0^u \|\boldsymbol{\alpha}'(v)\|\, \mathrm{d}v = \int_0^u \cosh v \, \mathrm{d}v = \sinh u \qquad \text{and} \qquad \boldsymbol{t}(u) = \frac{1}{\cosh u}(1, \sinh u),$$

so

$$\boldsymbol{\gamma}(u) = \boldsymbol{a}(u) - \sinh u \boldsymbol{t}(u) = \left(u - \frac{\sinh u}{\cosh u}, \cosh u - \frac{\sinh^2 u}{\cosh u}\right) = \frac{1}{\cosh u}(u\cosh u - \sinh u, 1)$$

Exchanging coordinate axes, we obtain a curve parametrized by

$$\widetilde{\boldsymbol{\gamma}}(u) = \frac{1}{\cosh u}(1, u\cosh u - \sinh u)$$

The tractrix from Exercise 1.6 is completely characterized by its property (d). Computing the corresponding distance for the curve $\widetilde{\boldsymbol{\gamma}}(u)$ we see that its trace is also a tractrix.

(b) As we have already computed in Exercise 2.1 and in (a),

$$\boldsymbol{t}(u) = \frac{1}{\cosh u}(1, \sinh u), \qquad \kappa(u) = \frac{1}{\cosh^2 u}$$

In particular, $\kappa(u)$ is never zero, and

$$\boldsymbol{n}(u) = \frac{1}{\cosh u}(-\sinh u, 1)$$

Now we can compute the evolute:

$$\boldsymbol{e}(u) = \boldsymbol{\alpha}(u) + \frac{1}{\kappa(u)}\boldsymbol{n}(u) = (u - \sinh u \cosh u, 2\cosh u)$$

as required.

(c) The singular points of $\boldsymbol{e}$ correspond to the vertices of $\boldsymbol{\alpha}$. We have

$$\kappa'(u) = \left(\frac{1}{\cosh^2 u}\right)' = -\frac{2\sinh u}{\cosh^3 u},$$

so $\kappa'(u) = 0$ if and only if $u = 0$. The only singular point of $\boldsymbol{e}$ is $(0, 2)$.

3.2. $(*)$ *Parallels.* Let $\boldsymbol{\alpha}$ be a plane curve parametrized by arc length, and let d be a real number. The curve $\boldsymbol{\beta}(u) = \boldsymbol{\alpha}(u) + d\boldsymbol{n}(u)$ is called the *parallel* to $\boldsymbol{\alpha}$ at distance d.

(a) Show that $\boldsymbol{\beta}$ is a regular curve except for values of u for which $d = 1/\kappa(u)$, where κ is the curvature of $\boldsymbol{\alpha}$.

(b) Show that the set of singular points of all the parallels (i.e., for all $d \in \mathbb{R}$) is the evolute of $\boldsymbol{\alpha}$.

Solution:

(a) Assume $\kappa(u) = 0$ or $d\kappa(u) \neq 1$. The latter is automatically satisfied if $\kappa(u) = 0$. So we just assume that $d\kappa(u) \neq 1$. We need to show that $\boldsymbol{\beta}'(u) \neq 0$. Since $\boldsymbol{\alpha}$ is unit speed, we have

$$\begin{aligned}\boldsymbol{\beta}'(u) = \boldsymbol{t}(u) + d\boldsymbol{n}'(u) = \boldsymbol{t}(u) + dA\boldsymbol{t}'(u) = \boldsymbol{t}(u) + d\kappa(u)A\boldsymbol{n}(u) = \\ = \boldsymbol{t}(u) + d\kappa(u)A^2\boldsymbol{t}(u) = \boldsymbol{t}(u) - d\kappa(u)\boldsymbol{t}(u) = (1 - d\kappa(u))\boldsymbol{t}(u),\end{aligned}$$

with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ are understood as columns. Note that $\|\boldsymbol{t}(u)\| = 1$, i.e., $\boldsymbol{t}(u) \neq 0$. The initial assumption implies that $(1 - d\kappa(u)) \neq 0$ and, therefore $\boldsymbol{\beta}'(u) \neq 0$, i.e., $\boldsymbol{\beta}(u)$ is regular.

In the case $\kappa(u) \neq 0$ and $d\kappa(u) = 1$, i.e., $d = 1/\kappa(u)$, we obviously have $\boldsymbol{\beta}'(u) = 0$, i.e., $\boldsymbol{\beta}(u)$ is singular.

(b) The evolute is only defined in the case that we have $\kappa(u) \neq 0$ for all u. So we assume this. We have seen in (a) that the singular points of the parallels are precisely those $\boldsymbol{\beta}(u)$ for which we have $d\kappa(u) = 1$, i.e., $d = 1/\kappa(u)$. This means that the set fo singular points of all parallels is

$$\{\boldsymbol{\alpha}(u) + d\boldsymbol{n}(u) \mid u \in I,\ d = 1/\kappa(u)\} = \{\boldsymbol{\alpha}(u) + \frac{1}{\kappa(u)}\boldsymbol{n}(u) \mid u \in I\}$$

which is precisely the parametrization of the evolute of $\boldsymbol{\alpha}$.

3.3. Let $\boldsymbol{\alpha}(u) : I \to \mathbb{R}^2$ be a smooth regular curve. Suppose there exists $u_0 \in I$ such that the distance $||\boldsymbol{\alpha}(u)||$ from the origin to the trace of $\boldsymbol{\alpha}$ is maximal at u_0. Show that the curvature $\kappa(u_0)$ of $\boldsymbol{\alpha}$ at u_0 satisfies

$$|\kappa(u_0)| \geq 1/||\boldsymbol{\alpha}(u_0)||$$

Solution:

Note first that the both sides of the inequality we want to prove do not depend on the parametrization, so we may assume without loss of generality that $\boldsymbol{\alpha}$ is parametrized by arc length.

Consider the function $f(u) = \|\boldsymbol{\alpha}\|^2$. Since $f(u)$ has a maximum at u_0, the first derivative of $f(u)$ at u_0 vanishes (cf. Exercise 1.4(b)), and the second derivative is non-positive. Thus, we have

$$0 \geq f''(u_0) = (\boldsymbol{\alpha}(u)\cdot\boldsymbol{\alpha}(u))''|_{u_0} = (2\boldsymbol{\alpha}'(u)\cdot\boldsymbol{\alpha}(u))'|_{u_0} = \boldsymbol{\alpha}''(u_0)\cdot\boldsymbol{\alpha}(u_0) + 2\|\boldsymbol{\alpha}'(u_0)\|^2 = \boldsymbol{\alpha}''(u_0)\cdot\boldsymbol{\alpha}(u_0) + 2$$

To satisfy the inequality above, we must have $\boldsymbol{\alpha}''(u_0)\cdot\boldsymbol{\alpha}(u_0) \leq -1$, which implies $|\boldsymbol{\alpha}''(u_0)\cdot\boldsymbol{\alpha}(u_0)| \geq 1$, and therefore

$$|\kappa(u_0)| = \|\boldsymbol{\alpha}''(u_0)\| \geq 1/\|\boldsymbol{\alpha}(u_0)\|$$

3.4. *Contact with circles.* The points $(x, y) \in \mathbb{R}^2$ of a circle are given as solutions of the equation $C(x, y) = 0$ where

$$C(x, y) = (x - a)^2 + (y - b)^2 - \lambda$$

Let $\boldsymbol{\alpha} = (x(u), y(u))$ be a plane curve. Suppose that the point $\boldsymbol{\alpha}(u_0)$ is also on some circle defined by $C(x, y)$. Then C vanishes at $(x(u_0), y(u_0))$ and the equation $g(u) = 0$ with

$$g(u) = C(x(u), y(u)) = (x(u) - a)^2 + (y(u) - b)^2 - \lambda$$

has a solution at u_0. If u_0 is a multiple solution of the equation, with $g^{(i)}(u_0) = 0$ for $i = 1, \ldots, k-1$ but $g^{(k)}(u_0) \neq 0$, we say that the curve $\boldsymbol{\alpha}$ and the circle have *k-point contact* at $\boldsymbol{\alpha}(u_0)$.

(a) Let a circle be tangent to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$. Show that $\boldsymbol{\alpha}$ and the circle have at least 2-point contact at $\boldsymbol{\alpha}(u_0)$.

(b) Suppose that $\kappa(u_0) \neq 0$. Show that $\boldsymbol{\alpha}$ and the circle have at least 3-point contact at $\boldsymbol{\alpha}(u_0)$ if and only if the center of the circle is the center of curvature of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$.

(c) Show that $\boldsymbol{\alpha}$ and the circle have at least 4-point contact if and only if the center of the circle is the center of curvature of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$ and $\boldsymbol{\alpha}(u_0)$ is a vertex of $\boldsymbol{\alpha}$.

Solution:

Denote by $\boldsymbol{c} = (a, b)$ the center of the circle $C(x, y) = 0$. Then the function $g(u) = C(x(u), y(u)) = (x(u) - a)^2 + (y(u) - b)^2 - \lambda$ can be written as

$$g(u) = (\boldsymbol{\alpha}(u) - \boldsymbol{c}) \cdot (\boldsymbol{\alpha}(u) - \boldsymbol{c}) - \lambda$$

(a) Differentiating $g(u)$, we obtain

$$g'(u) = 2(\boldsymbol{\alpha}(u) - \boldsymbol{c}) \cdot \boldsymbol{\alpha}'(u)$$

which vanishes if and only if $\boldsymbol{\alpha}'(u)$ is orthogonal to $\boldsymbol{\alpha}(u) - \boldsymbol{c}$. Note that $\boldsymbol{\alpha}(u) - \boldsymbol{c}$ is a radius of the circle, and the vector $\boldsymbol{\alpha}'(u)$ is orthogonal to a radius if and only if it is tangent to the circle.

(b) Differentiating $g'(u)$, we obtain

$$g''(u) = 2(\boldsymbol{\alpha}(u) - \boldsymbol{c}) \cdot \boldsymbol{\alpha}''(u) + 2\|\boldsymbol{\alpha}'(u)\|^2$$

Since $\boldsymbol{\alpha}(u) - \boldsymbol{c}$ is orthogonal to $\boldsymbol{\alpha}'(u)$, it is collinear with $\boldsymbol{\alpha}''(u)$, namely, it is equal to $\pm\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\boldsymbol{n}$. Assume $\kappa(u) > 0$ (if $\kappa(u) < 0$ the computations are similar), then $\boldsymbol{\alpha}''(u) = -\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\boldsymbol{n}$. Thus, $g''(u) = 0$ if and only if

$$-2\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\boldsymbol{n} \cdot \boldsymbol{\alpha}''(u) + 2\|\boldsymbol{\alpha}'(u)\|^2 = 0,$$

which is equivalent to

$$\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\| = \frac{\|\boldsymbol{\alpha}'(u)\|^2}{\boldsymbol{n} \cdot \boldsymbol{\alpha}''(u)}$$

The latter is equal to $1/\kappa(u)$ (see Exercise 2.2).

(c) Again, assume $\kappa(u) > 0$. According to (b), we can write

$$\begin{aligned} g''(u) &= -2\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\boldsymbol{n} \cdot \boldsymbol{\alpha}''(u) + 2\|\boldsymbol{\alpha}'(u)\|^2 = \\ &= -2\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\kappa(u)\|\boldsymbol{\alpha}'(u)\|^2 + 2\|\boldsymbol{\alpha}'(u)\|^2 = 2\|\boldsymbol{\alpha}'(u)\|^2(1 - \kappa(u)\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|) \end{aligned}$$

Differentiating this expression, we get

$$g'''(u) = 4\boldsymbol{\alpha}''(u) \cdot \boldsymbol{\alpha}'(u)(1 - \kappa(u)\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|) + 2\|\boldsymbol{\alpha}'(u)\|^2(-\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|'\kappa(u) - \|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|\kappa'(u))$$

Since the center $\boldsymbol{c}$ of the circle coincides with the center of curvature of $\boldsymbol{\alpha}$, the first summand iz equal to zero. The derivative of $\|\boldsymbol{\alpha}(u) - \boldsymbol{c}\|$ is also zero since $\boldsymbol{\alpha}'(u)$ is orthogonal to $\boldsymbol{\alpha}(u) - \boldsymbol{c}$ (cf. (a) or Exercise 1.4(b)). Thus, $g'''(u) = 0$ if and only if $\kappa'(u) = 0$, or, equivalently, $\boldsymbol{\alpha}(u)$ is a vertex of $\boldsymbol{\alpha}$.

4.1. Check that for two curves $\boldsymbol{\alpha}, \boldsymbol{\beta} : I \to \mathbb{R}^3$ holds

$$(\boldsymbol{\alpha}(u) \times \boldsymbol{\beta}(u))' = \boldsymbol{\alpha}'(u) \times \boldsymbol{\beta}(u) + \boldsymbol{\alpha}(u) \times \boldsymbol{\beta}'(u),$$

where $\boldsymbol{\alpha} \times \boldsymbol{\beta}$ is the cross-product in $\mathbb{R}^3$.

Solution: One can do a direct calculation in coordinates similar to Exercise 1.3. Alternatively, one can observe that coordinates of a cross-product are expressed via certain determinants which are miltilinear functions.

4.2. $(*)$ Find the curvature and torsion of the curve

$$\boldsymbol{\alpha}(u) = (au, bu^2, cu^3).$$

Solution:

We use Theorem 4.6. Since

$$\begin{aligned} \boldsymbol{\alpha}'(u) &= (a, 2bu, 3cu^2), \\ \boldsymbol{\alpha}''(u) &= (0, 2b, 6cu), \\ \boldsymbol{\alpha}'''(u) &= (0, 0, 6c), \end{aligned}$$

we have

$$\begin{aligned} \kappa(u) &= \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3} = \frac{\|(6bcu^2, -6acu, 2ab)\|}{(a^2 + 4b^2u^2 + 9c^2u^4)^{3/2}} = \frac{2(9b^2c^2u^4 + 9a^2c^2u^2 + a^2b^2)^{1/2}}{(a^2 + 4b^2u^2 + 9c^2u^4)^{3/2}}, \\ \tau(u) &= \frac{-(6bcu^2, -6acu, 2ab) \cdot (0, 0, 6c)}{4(9b^2c^2u^4 + 9a^2c^2u^2 + a^2b^2)} = \frac{-3abc}{(9b^2c^2u^4 + 9a^2c^2u^2 + a^2b^2)} \end{aligned}$$

4.3. $(*)$ Assume that $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular space curve parametrized by arc length.

(a) Determine all regular curves with vanishing curvature κ.

Hint: use Theorem 4.6

(b) Show that if the torsion τ of $\boldsymbol{\alpha}$ vanishes, then the trace of $\boldsymbol{\alpha}$ lies in a plane.

Hint: do NOT use Theorem 4.6

Solution:

(a) By Theorem 4.6, $\kappa(s) = 0$ if and only if $\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s) = 0$. Note that since $\boldsymbol{\alpha}$ is regular, $\boldsymbol{\alpha}'(s) \neq 0$.

If $\boldsymbol{\alpha}''(s) \equiv 0$, then $\boldsymbol{\alpha}'(s) = (a, b, c)$ for some constants $a, b, c \in \mathbb{R}$, and thus

$$\boldsymbol{\alpha}(s) = \boldsymbol{\alpha}_0 + s(a, b, c)$$

is a line.

Assume now that $\boldsymbol{\alpha}''(s) \neq 0$ at some point s (and thus, in some neighborhood of s). Then the unit normal $\boldsymbol{n}(s)$ is the unit vector defined by

$$\boldsymbol{n}(s) = \frac{\boldsymbol{\alpha}''(s)}{\|\boldsymbol{\alpha}''(s)\|},$$

so,

$$\kappa(s) = \frac{\|\boldsymbol{\alpha}''(s)\|}{\|\boldsymbol{n}(s)\|} = \|\boldsymbol{\alpha}''(s)\| \neq 0,$$

which leads to a contradiction.

Therefore, the only regular curve with zero curvature is a line.

(b) By Serret-Frenet equations, $\boldsymbol{b}' = \tau \boldsymbol{n}$. Thus, if $\tau \equiv 0$, then $\boldsymbol{b}$ is constant. In particular, all the planes spanned by $\boldsymbol{t}(s)$ and $\boldsymbol{n}(s)$ are parallel. We want to show that they all coincide.

Choose any s_0, and consider the function

$$f(s) = \boldsymbol{b} \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(s_0))$$

The derivative of this function is

$$f'(s) = \boldsymbol{b}' \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(s_0)) + \boldsymbol{b} \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(s_0))' = \boldsymbol{0} \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(s_0)) + \boldsymbol{b} \cdot \boldsymbol{\alpha}'(s) = 0$$

which implies that $f(s)$ is constant. Since for $f(s_0) = 0$, we see that $\boldsymbol{\alpha}(s)$ satisfies

$$\boldsymbol{b} \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}_0) = 0$$

for constant vectors $\boldsymbol{b}$ and $\boldsymbol{\alpha}_0$. The equation above is an equation of a plane in $\mathbb{R}^3$.

4.4. Assume that $\boldsymbol{\alpha}(s) = (x(s), y(s), 0)$, i.e., the trace of $\boldsymbol{\alpha}$ lies in the plane $z = 0$. Calculate the curvature κ of $\boldsymbol{\alpha}$ and its torsion τ. What is the relation of the curvature κ of the space curve $\boldsymbol{\alpha}$ and the (signed) curvature $\overline{\kappa}$ of the plane curve $\overline{\boldsymbol{\alpha}} : I \to \mathbb{R}^2$ defined by $\overline{\boldsymbol{\alpha}}(s) = (x(s), y(s))$ (i.e., the projection of the space curve $\boldsymbol{\alpha}$ to the plane $z = 0$)?

Solution:

Since $\boldsymbol{\alpha}$ lies in the plane $z = 0$, the tangent and normal vectors also lie in the plane, so the binormal vector is constant. Using the equation $\boldsymbol{b}' = \tau \boldsymbol{n}$ we see that $\tau \equiv 0$. The curvature of $\boldsymbol{\alpha}$ is clearly the absolute value of the curvature of $\overline{\boldsymbol{\alpha}}$.

4.5. Consider the regular curve given by

$$\boldsymbol{\alpha}(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}\right), \qquad s \in \mathbb{R},$$

where $a, b, c > 0$ and $c^2 = a^2 + b^2$. The curve $\boldsymbol{\alpha}$ is called a *helix*.

(a) Show that the trace of $\boldsymbol{\alpha}$ lies on the cylinder $x^2 + y^2 = a^2$.

(b) Show that $\boldsymbol{\alpha}$ is parametrized by arc length.

(c) Determine the curvature and torsion of $\boldsymbol{\alpha}$ (and notice that they are both constant).

(d) Determine the equation of the plane through $\boldsymbol{n}(s)$ and $\boldsymbol{t}(s)$ at each point of $\boldsymbol{\alpha}$ (this plane is called the *osculating plane*).

(e) Show that the line through $\boldsymbol{\alpha}(s)$ in direction $\boldsymbol{n}(s)$ meets the axis of the cylinder orthogonally.

(f) Show that the tangent lines to $\boldsymbol{\alpha}$ make a constant angle with the axis of the cylinder.

Solution:

(a)

$$x(s)^2 + y(s)^2 = a^2 \left(\cos^2 \frac{s}{c} + \sin^2 \frac{s}{c}\right) = a^2,$$

i.e., the trace of $\boldsymbol{\alpha}$ lies on the cylinder $x^2 + y^2 = a^2$.

(b) We have

$$\boldsymbol{\alpha}'(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right),$$

which implies

$$\|\boldsymbol{\alpha}'(s)\|^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1.$$

This shows that $\boldsymbol{\alpha}$ is unit speed.

(c) We have

$$\begin{aligned}
\boldsymbol{\alpha}''(s) &= \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right),\\
\boldsymbol{\alpha}'''(s) &= \left(\frac{a}{c^3}\sin\frac{s}{c}, -\frac{a}{c^3}\cos\frac{s}{c}, 0\right).
\end{aligned}$$

This implies

$$\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s) = \left(\frac{ab}{c^3}\sin\frac{s}{c}, -\frac{ab}{c^3}\cos\frac{s}{c}, \frac{a^2}{c^3}\right).$$

We conclude that

$$\|\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)\|^2 = \frac{a^2(a^2+b^2)}{c^6} = \frac{a^2}{c^4},$$

i.e.,

$$\kappa = \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3} = \frac{a}{c^2}.$$

Moreover, we have

$$(\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)) \cdot \boldsymbol{\alpha}'''(s) = \frac{a^2 b}{c^6}\sin^2\frac{s}{c} + \frac{a^2 b}{c^6}\cos^2\frac{s}{c} = \frac{a^2 b}{c^6}.$$

This implies that

$$\tau = -\frac{a^2 b}{c^6} \cdot \frac{c^4}{a^2} = -\frac{b}{c^2}.$$

(d) The osculating plane is orthogonal to the binormal vector $\boldsymbol{b}(s)$, and thus to $\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)$ which is collinear to $\boldsymbol{b}(s)$. We have already computed in (c) that

$$\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s) = \left(\frac{ab}{c^3}\sin\frac{s}{c}, -\frac{ab}{c^3}\cos\frac{s}{c}, \frac{a^2}{c^3}\right).$$

Therefore, the equation of the osculating plane at $\boldsymbol{\alpha}(s) = (x(s), y(s), z(s))$ can be written as

$$\frac{ab}{c^3}\sin\frac{s}{c}(x - x(s)) - \frac{ab}{c^3}\cos\frac{s}{c}(y - y(s)) + \frac{a^2}{c^3}(z - z(s)) = 0.$$

After plugging in the explicit expressions for $\boldsymbol{\alpha}(s)$ and multiplying by c^3/a we obtain

$$xb\sin\frac{s}{c} - yb\cos\frac{s}{c} + az - ab\frac{s}{c} = 0$$

(e) Normalizing the expression for $\boldsymbol{\alpha}''(s)$ obtained in (c), we see that $\boldsymbol{n}(s) = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right)$. Since the z-coordinate of $\boldsymbol{n}(s)$ is zero, $\boldsymbol{n}(s)$ is orthogonal to the z-axis (which is also the axis of the cylinder). Note also that $a\boldsymbol{n}(s)$ is a projection of $-\boldsymbol{\alpha}(s)$ onto the horizontal plane, so the line $\boldsymbol{\alpha}(s) + u\boldsymbol{n}(s)$ meets the z-axis at $u = a$.

(f) To find the cosine of the angle, we need to compute the dot product of the unit tangent vector and the unit vector in the direction of the axis of the cylinder. The latter has coordinates $(0, 0, 1)$, so the cosine is equal to

$$(0,0,1) \cdot \left(-\frac{a}{c} \sin \frac{s}{c}, \frac{a}{c} \cos \frac{s}{c}, \frac{b}{c} \right) = \frac{b}{c}$$

which is constant.