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Differential Geometry 24.10.2019

Homework 3-4
Starred problems due on Thursday, 7 November.

Evolute and involute

3.1. Let $\boldsymbol{\alpha}$ denote the catenary from Exercise 2.1. Show that

(a) the involute of $\boldsymbol{\alpha}$ starting from $(0,1)$ is the tractrix from Exercise 1.6 (with x- and y-axes exchanged and different parametrization);

(b) the evolute of $\boldsymbol{\alpha}$ is the curve given by

$$\boldsymbol{\beta}(u) = (u - \sinh u \cosh u, 2\cosh u)$$

(c) Find the singular points of $\boldsymbol{\beta}$ and give a sketch of its trace.

3.2. $(*)$ *Parallels.* Let $\boldsymbol{\alpha}$ be a plane curve parametrized by arc length, and let d be a real number. The curve $\boldsymbol{\beta}(u) = \boldsymbol{\alpha}(u) + d\boldsymbol{n}(u)$ is called the *parallel* to $\boldsymbol{\alpha}$ at distance d.

(a) Show that $\boldsymbol{\beta}$ is a regular curve except for values of u for which $d = 1/\kappa(u)$, where κ is the curvature of $\boldsymbol{\alpha}$.

(b) Show that the set of singular points of all the parallels (i.e., for all $d \in \mathbb{R}$) is the evolute of $\boldsymbol{\alpha}$.

3.3. Let $\boldsymbol{\alpha}(u) : I \to \mathbb{R}^2$ be a smooth regular curve. Suppose there exists $u_0 \in I$ such that the distance $||\boldsymbol{\alpha}(u)||$ from the origin to the trace of $\boldsymbol{\alpha}$ is maximal at u_0. Show that the curvature $\kappa(u_0)$ of $\boldsymbol{\alpha}$ at u_0 satisfies

$$|\kappa(u_0)| \geq 1/||\boldsymbol{\alpha}(u_0)||$$

3.4. *Contact with circles.* The points $(x, y) \in \mathbb{R}^2$ of a circle are given as solutions of the equation $C(x, y) = 0$ where

$$C(x, y) = (x - a)^2 + (y - b)^2 - \lambda$$

Let $\boldsymbol{\alpha} = (x(u), y(u))$ be a plane curve. Suppose that the point $\boldsymbol{\alpha}(u_0)$ is also on some circle defined by $C(x, y)$. Then C vanishes at $(x(u_0), y(u_0))$ and the equation $g(u) = 0$ with

$$g(u) = C(x(u), y(u)) = (x(u) - a)^2 + (y(u) - b)^2 - \lambda$$

has a solution at u_0. If u_0 is a multiple solution of the equation, with $g^{(i)}(u_0) = 0$ for $i = 1, \ldots, k-1$ but $g^{(k)}(u_0) \neq 0$, we say that the curve $\boldsymbol{\alpha}$ and the circle have *k-point contact* at $\boldsymbol{\alpha}(u_0)$.

(a) Let a circle be tangent to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$. Show that $\boldsymbol{\alpha}$ and the circle have at least 2-point contact at $\boldsymbol{\alpha}(u_0)$.

(b) Suppose that $\kappa(u_0) \neq 0$. Show that $\boldsymbol{\alpha}$ and the circle have at least 3-point contact at $\boldsymbol{\alpha}(u_0)$ if and only if the centre of the circle is the centre of curvature of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$.

(c) Show that $\boldsymbol{\alpha}$ and the circle have at least 4-point contact if and only if the centre of the circle is the centre of curvature of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$ and $\boldsymbol{\alpha}(u_0)$ is a vertex of $\boldsymbol{\alpha}$.

Space curves - 1

4.1. Check that for two curves $\boldsymbol{\alpha}, \boldsymbol{\beta} : I \to \mathbb{R}^3$ holds

$$(\boldsymbol{\alpha}(u) \times \boldsymbol{\beta}(u))' = \boldsymbol{\alpha}'(u) \times \boldsymbol{\beta}(u) + \boldsymbol{\alpha}(u) \times \boldsymbol{\beta}'(u),$$

where $\boldsymbol{\alpha} \times \boldsymbol{\beta}$ is the cross-product in $\mathbb{R}^3$.

4.2. $(*)$ Find the curvature and torsion of the curve

$$\boldsymbol{\alpha}(u) = (au, bu^2, cu^3).$$

4.3. $(*)$ Assume that $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular space curve parametrized by arc length.

(a) Determine all regular curves with vanishing curvature κ.

Hint: use Theorem 4.6

(b) Show that if the torsion τ of $\boldsymbol{\alpha}$ vanishes, then the trace of $\boldsymbol{\alpha}$ lies in a plane.

Hint: do NOT use Theorem 4.6

4.4. Assume that $\boldsymbol{\alpha}(s) = (x(s), y(s), 0)$, i.e., the trace of $\boldsymbol{\alpha}$ lies in the plane $z = 0$. Calculate the curvature κ of $\boldsymbol{\alpha}$ and its torsion τ. What is the relation of the curvature κ of the space curve $\boldsymbol{\alpha}$ and the (signed) curvature $\overline{\kappa}$ of the plane curve $\overline{\boldsymbol{\alpha}} : I \to \mathbb{R}^2$ defined by $\overline{\boldsymbol{\alpha}}(s) = (x(s), y(s))$ (i.e., the projection of the space curve $\boldsymbol{\alpha}$ to the plane $z = 0$)?

4.5. Consider the regular curve given by

$$\boldsymbol{\alpha}(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}\right), \qquad s \in \mathbb{R},$$

where $a, b, c > 0$ and $c^2 = a^2 + b^2$. The curve $\boldsymbol{\alpha}$ is called a *helix*.

(a) Show that the trace of $\boldsymbol{\alpha}$ lies on the cylinder $x^2 + y^2 = a^2$.

(b) Show that $\boldsymbol{\alpha}$ is parametrized by arc length.

(c) Determine the curvature and torsion of $\boldsymbol{\alpha}$ (and notice that they are both constant).

(d) Determine the equation of the plane through $\boldsymbol{n}(s)$ and $\boldsymbol{t}(s)$ at each point of $\boldsymbol{\alpha}$ (this plane is called the *osculating plane*).

(e) Show that the line through $\boldsymbol{\alpha}(s)$ in direction $\boldsymbol{n}(s)$ meets the axis of the cylinder orthogonally.

(f) Show that the tangent lines to $\boldsymbol{\alpha}$ make a constant angle with the axis of the cylinder.

Remark: In fact, a helix can be characterized by (a) and (f). If we drop (a), then we obtain a *generalized helix* (see next homework). Another way how to characterize a helix is by (c), i.e., the fact that the curvature and torsion are constant. *Why?*