Durham University
Anna Felikson

Differential Geometry 26.11.2019

Solutions 5-6

5.1. ($*$) A curve $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is called a *(generalized) helix* if its tangent lines make a constant angle with a fixed direction in $\mathbb{R}^3$.

(a) Prove that the curve

$$\boldsymbol{\alpha}(s) = \Big(\frac{a}{c}\int_{s_0}^{s} \sin\vartheta(v)\,\mathrm{d}v, \frac{a}{c}\int_{s_0}^{s} \cos\vartheta(v)\,\mathrm{d}v, \frac{b}{c}s\Big),$$

with $s_0 \in I$, $c^2 = a^2 + b^2$, $a \neq 0$, $b \neq 0$ and $\vartheta'(s) > 0$ is a (generalized) helix.

(b) Assume that $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular curve with $\tau(s) \neq 0$ for all $s \in I$. Prove that $\boldsymbol{\alpha}$ is a (generalized) helix if and only if κ/τ is constant.

Solution:

(a) We have

$$\boldsymbol{t} = \boldsymbol{\alpha}'(s) = \left(\frac{a}{c}\sin\vartheta(s),\ \frac{a}{c}\cos\vartheta(s),\ \frac{b}{c}\right),$$

so $\|\boldsymbol{t}\| = 1$, that is $\boldsymbol{\alpha}$ is parametrized by arc length.

One way to show that $\boldsymbol{\alpha}$ is a (generalized) helix is to use (b). For this, we compute

$$\boldsymbol{t}' = \boldsymbol{\alpha}'(s) = \Big(\frac{a}{c}\vartheta'(s)\cos\vartheta(s),\ -\frac{a}{c}\vartheta'(s)\sin\vartheta(s), 0\Big) = \frac{a}{c}\vartheta'(s)\left(\cos\vartheta(s), -\sin\vartheta(s), 0\right).$$

We may assume without loss of generality that $\frac{a}{c}\vartheta'(s) > 0$ and take $\kappa(s) = \frac{a}{c}\vartheta'(s)$ and $\boldsymbol{n} = (\cos\vartheta(s), -\sin\vartheta(s), 0)$. Then

$$\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n} = \left(\frac{b}{c}\sin\vartheta(s),\ \frac{b}{c}\cos\vartheta(s), -\frac{a}{c}\right),$$

and

$$\boldsymbol{b}' = \left(\frac{b}{c}\vartheta'(s)\cos\vartheta(s),\ -\frac{b}{c}\sin\vartheta(s), 0\right) = \frac{b}{c}\vartheta'(s)\boldsymbol{n}.$$

Hence $\tau = \frac{b}{c}\vartheta'(s)$ and $\kappa/\tau = \frac{a}{b}$ is constant. It follows from part (b) that $\boldsymbol{\alpha}$ is a generalized helix.

A much simpler way to solve the problem is to guess the vector $\boldsymbol{v}$ such that $\boldsymbol{t} \cdot \boldsymbol{v}$ is constant. Indeed, one can see that z-coordinate of $\boldsymbol{t}$ is equal to b/c, i.e. it is constant. Thus, $\boldsymbol{t}$ makes a constant angle with vector $(0, 0, 1)$, i.e. with z-axis.

(b) We may assume that $\boldsymbol{\alpha}$ is parametrized by arc length. By definition, $\boldsymbol{\alpha}$ is a (generalized) helix if and only if there exists a constant vector $\boldsymbol{v}$ such that

$$\frac{\boldsymbol{t} \cdot \boldsymbol{v}}{\|\boldsymbol{t}\|\,\|\boldsymbol{v}\|} = \frac{\boldsymbol{t} \cdot \boldsymbol{v}}{\|\boldsymbol{v}\|} = \text{const}$$

We may assume that $\boldsymbol{v}$ has unit length, so the equality above is equivalent to

$$\boldsymbol{t} \cdot \boldsymbol{v} = \text{const}$$

Equivalently, $\boldsymbol{\alpha}$ is a (generalized) helix if and only if there exists a constant vector $\boldsymbol{v}$ such that

$$\boldsymbol{t}' \cdot \boldsymbol{v} = 0 \iff \boldsymbol{n} \cdot \boldsymbol{v} = 0 \iff \boldsymbol{v} = c\boldsymbol{t} + d\boldsymbol{b}.$$

Since $\boldsymbol{v}$ has unit length, we have $c^2 + d^2 = 1$. Then $\boldsymbol{v}$ makes a constant angle with $\boldsymbol{t}$ if and only if $c = \text{const}$ The vector $\boldsymbol{v}$ is a constant vector if and only if $(c\boldsymbol{t} + d\boldsymbol{b})' = 0$, that is if and only if

$$c'\boldsymbol{t} + c\boldsymbol{t}' + d'\boldsymbol{b} + d\boldsymbol{b}' = c\kappa\boldsymbol{n} + d'\boldsymbol{b} + d\tau\boldsymbol{n} = d'\boldsymbol{b} + (c\kappa + d\tau)\boldsymbol{n} = 0,$$

which holds if and only if

$$d' = c\kappa + d\tau = 0,$$

if and only if $\kappa/\tau = -d/c = \text{const}$

5.2. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be regular curves in $\mathbb{R}^3$ such that, for each u, the principal normals $\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$ and $\boldsymbol{n}_{\boldsymbol{\beta}}(u)$ are parallel. Prove that the angle between $\boldsymbol{t}_{\boldsymbol{\alpha}}(u)$ and $\boldsymbol{t}_{\boldsymbol{\beta}}(u)$ is independent of u. Prove also that if the line through $\boldsymbol{\alpha}(u)$ in direction $\boldsymbol{n}_{\boldsymbol{\alpha}(u)}$ coincides with the line through $\boldsymbol{\beta}(u)$ in direction $\boldsymbol{n}_{\boldsymbol{\beta}(u)}$ then

$$\boldsymbol{\beta}(u) = \boldsymbol{\alpha}(u) + r\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$$

for some real number r.

Solution:

We may assume that one of the curves (say, $\boldsymbol{\alpha}$) is parametrized by arc length. Let

$$f(u) = \boldsymbol{t}_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\beta}}(u)$$

We want to show that $f'(u) \equiv 0$.

$$\begin{aligned} f'(u) = \boldsymbol{t}_{\boldsymbol{\alpha}}'(u) \cdot \boldsymbol{t}_{\boldsymbol{\beta}}(u) + \boldsymbol{t}_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\beta}}'(u) = \kappa_{\boldsymbol{\alpha}}(u)\boldsymbol{n}_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\beta}}(u) + \boldsymbol{t}_{\boldsymbol{\alpha}}(u) \cdot \|\boldsymbol{\beta}'(u)\|\kappa_{\boldsymbol{\beta}}(u)\boldsymbol{n}_{\boldsymbol{\beta}}(u) = \\ = \boldsymbol{n}_{\boldsymbol{\alpha}}(u) \cdot (\kappa_{\boldsymbol{\alpha}}(u)\boldsymbol{t}_{\boldsymbol{\beta}}(u) + \lambda(u)\|\boldsymbol{\beta}'(u)\|\kappa_{\boldsymbol{\beta}}\boldsymbol{t}_{\boldsymbol{\alpha}}(u)) \end{aligned}$$

for the function $\lambda(u)$ defined by $\boldsymbol{n}_{\boldsymbol{\beta}}(u) = \lambda(u)\boldsymbol{n}_{\boldsymbol{\alpha}}$. Now, $\boldsymbol{n}_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\alpha}}(u) = 0$, and

$$\boldsymbol{n}_{\boldsymbol{\alpha}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\beta}}(u) = \lambda^{-1}(u)\boldsymbol{n}_{\boldsymbol{\beta}}(u) \cdot \boldsymbol{t}_{\boldsymbol{\beta}}(u) = 0,$$

so $f'(u) \equiv 0$.

Now assume the lines $\{\boldsymbol{\alpha}(u) + \mu_1\boldsymbol{n}_{\boldsymbol{\alpha}}(u) \,|\, \mu_1 \in \mathbb{R}\}$ and $\{\boldsymbol{\beta}(u) + \mu_2\boldsymbol{n}_{\boldsymbol{\beta}}(u) \,|\, \mu_2 \in \mathbb{R}\}$ coincide, i.e.

$$\boldsymbol{\alpha}(u) - \boldsymbol{\beta}(u) = \mu(u)\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$$

for some $\mu(u) \in \mathbb{R}$. We want to show that $\mu(u)$ is constant. We can write

$$\mu(u) = \boldsymbol{n}_{\boldsymbol{\alpha}}(u) \cdot (\boldsymbol{\alpha}(u) - \boldsymbol{\beta}(u)),$$

therefore

$$\mu'(u) = \boldsymbol{n}_{\boldsymbol{\alpha}}'(u) \cdot (\boldsymbol{\alpha}(u) - \boldsymbol{\beta}(u)) + n_{\boldsymbol{\alpha}}(u) \cdot (\boldsymbol{t}_{\boldsymbol{\alpha}}(u) - \boldsymbol{t}_{\boldsymbol{\beta}}(u))$$

The first summand vanishes since $\boldsymbol{\alpha}(u) - \boldsymbol{\beta}(u) = \mu(u)n_{\boldsymbol{\alpha}}(u)$ is parallel to $n_{\boldsymbol{\alpha}}(u)$, and $\boldsymbol{n}_{\boldsymbol{\alpha}}'(u) \cdot \boldsymbol{n}_{\boldsymbol{\alpha}}(u) = 0$. The second summand vanishes since $\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$ is parallel to $\boldsymbol{n}_{\boldsymbol{\beta}}(u)$.

5.3. ($*$) Let $\boldsymbol{\alpha}$ be the curve in $\mathbb{R}^3$ given by

$$\boldsymbol{\alpha}(u) = e^u(\cos u, \sin u, 1), \qquad u \in \mathbb{R}.$$

If $0 < \lambda_0 < \lambda_1$, find the length of the segment of $\boldsymbol{\alpha}$ which lies between the planes $z = \lambda_0$ and $z = \lambda_1$. Show also that the curvature and torsion of $\boldsymbol{\alpha}$ are both inversely proportional to e^u.

Solution:

We have

$$\boldsymbol{\alpha}'(u) = e^u(\cos u, \sin u, 1) + e^u(-\sin u, \cos u, 0) = (e^u \cos u - e^u \sin u, e^u \sin u + e^u \cos u, e^u),$$

$$\|\boldsymbol{\alpha}'(u)\| = e^u\sqrt{(\cos u - \sin u)^2 + (\sin u + \cos u)^2 + 1} = e^u\sqrt{3}.$$

We first need to find the parameter values when $\boldsymbol{\alpha}$ intersects the planes $z = \lambda_0$ and $z = \lambda_1$. The z-component of $\boldsymbol{\alpha}(u)$ is e^u, so $e^u = \lambda$ implies $u = \ln \lambda$. Then the arc length ℓ between where the curve intersects the planes $z = \lambda_0$ and $z = \lambda_1$ with $0 < \lambda_0 < \lambda_1$ is given by integrating $\|\boldsymbol{\alpha}'(u)\|$ between the corresponding parameter values, namely $u_0 = \ln \lambda_0$ and $u_1 = \ln \lambda_1$. So

$$\ell = \int_{u_0}^{u_1} \|\boldsymbol{\alpha}'(u)\| \,\mathrm{d}u = \int_{u_0}^{u_1} \sqrt{3}e^u \,\mathrm{d}u = \sqrt{3}\big[e^u\big]_{u_0}^{u_1} = \sqrt{3}(e^{u_1} - e^{u_0}) = \sqrt{3}(\lambda_1 - \lambda_0).$$

To compute the curvature we use the formula

$$\kappa = \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3}.$$

As a result, we obtain

$$\kappa(u) = \frac{\sqrt{2}}{3} \cdot e^{-u}$$

which has the desired form

$$\text{const} \cdot \frac{1}{e^u}.$$

Now one can note that $\boldsymbol{\alpha}$ is a generalized helix: indeed, the cosine of the angle formed by $\boldsymbol{\alpha}'(u)$ with vector $(0, 0, 1)$ is

$$\frac{(e^u \cos u - e^u \sin u, e^u \sin u + e^u \cos u, e^u) \cdot (0, 0, 1)}{\sqrt{3}e^u} = \frac{1}{\sqrt{3}}$$

which is constant. Thus, by Exercise 5.1, the torsion is also proportional to $1/e^u$.

Alternatively, one can compute the torsion explicitly to see that

$$\tau(u) = -\frac{1}{3} \cdot e^{-u}$$

which is also of required form.

5.4. Let $\boldsymbol{\alpha}$ be a curve parametrized by arc length with nowhere vanishing curvature κ and torsion τ. Show that if the trace of $\boldsymbol{\alpha}$ lies on a sphere then

$$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'.$$

Is the converse true?

Solution: Suppose that $\boldsymbol{\alpha}$ lies on the sphere with centre $\boldsymbol{c}$ and radius r. Then

$$(\boldsymbol{\alpha}-\boldsymbol{c})\cdot(\boldsymbol{\alpha}-\boldsymbol{c}) = r^2 \tag{$*$}$$

Differentiating $(*)$ once we get

$$\boldsymbol{t}\cdot(\boldsymbol{\alpha}-\boldsymbol{c}) = 0.$$

This means that there exist $x, y \in \mathbb{R}$ such that

$$\boldsymbol{\alpha}-\boldsymbol{c} = x\boldsymbol{n} + y\boldsymbol{b}.$$

Differentiating the equality above we obtain

$$\boldsymbol{t} = x'\boldsymbol{n} + x\boldsymbol{n}' + y'\boldsymbol{b} + y\boldsymbol{b}' = x'\boldsymbol{n} + x(-\kappa\boldsymbol{t} - \tau\boldsymbol{b}) + y'\boldsymbol{b} + y\tau\boldsymbol{n} = -x\kappa\boldsymbol{t} + (x' + y\tau)\boldsymbol{n} + (-x\tau + y')\boldsymbol{b}$$

In particular, this implies that

$$-x\tau + y' = 0$$

Let us find x and y. Differentiating $(*)$ twice we get

$$\kappa\boldsymbol{n}\cdot(\boldsymbol{\alpha}-\boldsymbol{c}) + 1 = 0 \tag{$**$}$$

Thus,

$$\kappa x + 1 = 0 \iff x = -\frac{1}{\kappa}.$$

Differentiating $(**)$ we get

$$\kappa'\boldsymbol{n}\cdot(\boldsymbol{\alpha}-\boldsymbol{c}) + \kappa(-\kappa\boldsymbol{t} - \tau\boldsymbol{b})\cdot(\boldsymbol{\alpha}-\boldsymbol{c}) + \kappa\boldsymbol{n}\cdot\boldsymbol{t} = 0$$

Since $\boldsymbol{n}\cdot\boldsymbol{t} = 0$, this implies

$$\kappa'\boldsymbol{n}\cdot(-\frac{1}{\kappa}\boldsymbol{n} + y\boldsymbol{b}) + \kappa(-\kappa\boldsymbol{t} - \tau\boldsymbol{b})\cdot(-\frac{1}{\kappa}\boldsymbol{n} + y\boldsymbol{b}) = 0,$$

which gives

$$-\frac{\kappa'}{\kappa} - \kappa\tau y = 0.$$

Hence,

$$y = -\frac{\kappa'}{\kappa^2\tau}$$

Now the equality $-x\tau + y' = 0$ obtained above becomes

$$\frac{\tau}{\kappa} = \Big(\frac{\kappa'}{\tau\kappa^2}\Big)'.$$

The converse is also true (see e.g. the solution of Exercise 5.5.(b))

5.5. Let $\boldsymbol{\alpha}$ be a regular curve parametrized by arc length with $\kappa > 0$ and $\tau \neq 0$. Denote by $\boldsymbol{n}$ and $\boldsymbol{b}$ the principal normal and the binormal of $\boldsymbol{\alpha}$.

(a) If $\boldsymbol{\alpha}$ lies on a sphere with center $\boldsymbol{c} \in \mathbb{R}^3$ and radius $r > 0$, show that

$$\boldsymbol{\alpha} - \boldsymbol{c} = -\rho\boldsymbol{n} - \rho'\sigma\boldsymbol{b},$$

where $\rho = 1/\kappa$ and $\sigma = -1/\tau$. Deduce that $r^2 = \rho^2 + (\rho'\sigma)^2$.

(b) Conversely, if $\rho^2 + (\rho'\sigma)^2$ has constant value r^2 and $\rho' \neq 0$, show that $\boldsymbol{\alpha}$ lies on a sphere of radius r.

Hint: Show that the curve $\boldsymbol{\alpha} + \rho\boldsymbol{n} + \rho'\sigma\boldsymbol{b}$ is constant.

Solution:

(a) Suppose that $\boldsymbol{\alpha}$ lies on the sphere with center $\boldsymbol{c}$ and radius r. From the solution of Exercise 5.4 we know that

$$\boldsymbol{\alpha} - \boldsymbol{c} = x\boldsymbol{n} + y\boldsymbol{b},$$

where

$$x = -\frac{1}{\kappa}, \qquad y = -\frac{\kappa'}{\kappa^2\tau}$$

We have thus

$$\boldsymbol{\alpha} - \boldsymbol{c} = -\frac{1}{\kappa}\boldsymbol{n} - \frac{\kappa'}{\kappa^2\tau}\boldsymbol{b} = -\rho\boldsymbol{n} - \rho'\sigma\boldsymbol{b},$$

where $\rho = 1/\kappa$ and $\sigma = -1/\tau$. Now,

$$r^2 = (\boldsymbol{\alpha} - \boldsymbol{c}) \cdot (\boldsymbol{\alpha} - \boldsymbol{c}) = (-\rho\boldsymbol{n} - \rho'\sigma\boldsymbol{b}) \cdot (-\rho\boldsymbol{n} - \rho'\sigma\boldsymbol{b}) = \rho^2 + (\rho'\sigma)^2.$$

(b) Suppose that $\rho^2 + (\rho'\sigma)^2 = r^2$. Differentiating we get

$$\rho'(\rho + (\rho'\sigma)'\sigma) = 0.$$

As $\rho' \neq 0$, it follows that

$$\rho + (\rho'\sigma)'\sigma = 0$$

or equivalently,

$$-\rho\tau + (\rho'\sigma)' = 0$$

The curve $\boldsymbol{\alpha} + \rho\boldsymbol{n} + \rho'\sigma\boldsymbol{b}$ is constant (i.e, is a point) if and only if $(\boldsymbol{\alpha} + \rho\boldsymbol{n} + \rho'\sigma\boldsymbol{b})' = 0$. We have,

$$\begin{aligned}
(\boldsymbol{\alpha} + \rho\boldsymbol{n} + \rho'\sigma\boldsymbol{b})' &= \boldsymbol{t} + \rho'\boldsymbol{n} + \rho\boldsymbol{n}' + (\rho'\sigma)'\boldsymbol{b} + (\rho'\sigma)\boldsymbol{b}' \\
&= \boldsymbol{t} + \rho'\boldsymbol{n} + \rho(-\kappa\boldsymbol{t} - \tau\boldsymbol{b}) + (\rho'\sigma)'\boldsymbol{b} + (\rho'\sigma)\tau\boldsymbol{n} \\
&= (1 - \rho\kappa)\boldsymbol{t} + (\rho' + \rho'\sigma\tau)\boldsymbol{n} + (-\tau\rho + (\rho'\sigma)')\boldsymbol{b} \\
&= 0\boldsymbol{t} + 0\boldsymbol{n} + 0\boldsymbol{b} \\
&= 0.
\end{aligned}$$

We conclude that $\boldsymbol{\alpha} + \rho\boldsymbol{n} + \rho'\sigma\boldsymbol{b} = \boldsymbol{c}$, for some point $\boldsymbol{c}$. Then

$$\boldsymbol{\alpha} - \boldsymbol{c} = -\rho\boldsymbol{n} - \rho'\sigma\boldsymbol{b}$$

and as $\rho^2 + (\rho'\sigma)^2$ has constant value r^2, $(\boldsymbol{\alpha} - \boldsymbol{c}) \cdot (\boldsymbol{\alpha} - \boldsymbol{c}) = r^2$. This means that the curve $\boldsymbol{\alpha}$ lies on the sphere with center $\boldsymbol{c}$ and radius r.

6.1. Let $U \subset \mathbb{R}^2$ be an open set. Show that the set

$$\{(x, y, z) \in \mathbb{R}^3 \,|\, z = 0 \text{ and } (x, y) \in U\}$$

is a regular surface.

Solution:

Take $V = \mathbb{R}^3$ and $\boldsymbol{x}(u, v) = (u, v, 0)$. Then all the assumtions of the definition of a surface hold immediately. One could also note that this set is a regular surface as a graph of the zero function.

6.2. Stereographic projection
Let $S^2(1) = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ be a 2-dimensional unit sphere. For $(u, v) \in \mathbb{R}^2$, let $\boldsymbol{x}(u, v)$ be the point of intersection of the line in $\mathbb{R}^3$ through $(u, v, 0)$ and $(0, 0, 1)$ with $S^2(1)$ (different from $(0, 0, 1)$).

(a) Find an explicit formula for $\boldsymbol{x}(u, v)$.

(b) Let P be the plane given by $\{z = 1\}$, and for $(x, y, z) \in \mathbb{R}^3 \setminus P$, let $\boldsymbol{F}(x, y, z) \in \mathbb{R}^2$ be such that $(\boldsymbol{F}(x, y, z), 0) \in \mathbb{R}^3$ is the intersection with the (x, y)-plane of the line through $(0, 0, 1)$ and (x, y, z). Show that

$$\boldsymbol{F}(x, y, z) = \frac{1}{1 - z}(x, y).$$

(c) Show that $\boldsymbol{F} \circ \boldsymbol{x} = \mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$ and deduce that $\boldsymbol{x}$ is a local parametrization of $S^2(1) \setminus \{(0, 0, 1)\}$.

Solution:

(a) We are looking for the intersection of the line L through $(0, 0, 1)$ and $(u, v, 0)$, $(u, v) \in \mathbb{R}^2$, with $S^2(1)$. This line can be parametrized by

$$\boldsymbol{L}(t) := (0, 0, 1) + t\big((u, v, 0) - (0, 0, 1)\big) = (tu, tv, 1 - t), \qquad t \in \mathbb{R}.$$

Now find $t \in \mathbb{R}$ such that $\boldsymbol{L}(t) \in S^2(1)$, i.e., that

$$(tu)^2 + (tv)^2 + (1 - t)^2 = 1.$$

This equation is equivalent to

$$t\big(t(u^2 + v^2 + 1) - 2\big) = 0,$$

hence

$$t = 0 \qquad \text{or} \qquad t = \frac{2}{u^2 + v^2 + 1}.$$

The former solution gives $\boldsymbol{L}(0) = (0, 0, 1)$, and we reject this point. We are looking for the other solution on the sphere, namely

$$\boldsymbol{L}(t) = \Big(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, 1 - \frac{2}{u^2 + v^2 + 1}\Big) = \frac{1}{u^2 + v^2 + 1}\big(2u, 2v, u^2 + v^2 - 1\big).$$

Therefore, $\boldsymbol{x} : U = \mathbb{R}^2 \to S = S^2(1)$ is given by

$$\boldsymbol{x}(u, v) = \frac{1}{u^2 + v^2 + 1}\big(2u, 2v, u^2 + v^2 - 1\big).$$

(b) Let $\boldsymbol{l}$ be the line through $(0,0,1)$ and (x,y,z) where $(x,y,z) \in \mathbb{R}^3 \setminus P$, i.e., $z \neq 1$. Then $\boldsymbol{l}$ can be parametrized by

$$\boldsymbol{l}(t) := (0,0,1) + t\big((x,y,z) - (0,0,1)\big) = (tx, ty, t(z-1)+1), \qquad t \in \mathbb{R}.$$

Its intersection with the xy-plane yields the condition

$$t(z-1)+1 = 0, \quad \text{i.e.,} \quad t = \frac{1}{1-z}$$

on the parameter t (note that $z \neq 1$). Hence the point in the xy-plane is

$$\boldsymbol{l}(1/(1-z)) = \frac{1}{1-z}(x,y,0).$$

In particular, $\boldsymbol{F}$ has the form

$$\boldsymbol{F}(x,y,z) = \Big(\frac{x}{1-z}, \frac{y}{1-z}\Big)$$

given by the first two coordinates of $\boldsymbol{l}(1/(1-z))$.

(c) We have

$$\begin{aligned}(\boldsymbol{F} \circ \boldsymbol{x})(u,v) &= \boldsymbol{F}(\boldsymbol{x}(u,v)) \\ &= \boldsymbol{F}\Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \underbrace{1 - \frac{2}{u^2+v^2+1}}_{z}\Big) \\ &= (u,v)\end{aligned}$$

since $1/(1-z) = (u^2+v^2+1)/2$, hence $\boldsymbol{F} \circ \boldsymbol{x}$ is the identity map on $U = \mathbb{R}^2$.

We conclude now as follows:

(i) $\boldsymbol{x}$ is a smooth map on $U = \mathbb{R}^2$, since its components are rational functions and the denominator never vanishes. Moreover,

$$\boldsymbol{x} : U = \mathbb{R}^2 \to \boldsymbol{x}(U) = S^2(1) \setminus \{(0,0,1)\}, \tag{$*$}$$

hence we may choose $V = \mathbb{R}^3 \setminus P$ in the definition of a regular surface (or any other open set containing $S^2(1) \setminus \{(0,0,1)\}$ like $V = \mathbb{R}^2 \times (-\infty, 1)$)

(ii) From $\boldsymbol{F} \circ \boldsymbol{x} = \mathrm{id}_U$ we conclude that $\boldsymbol{x}$ is bijective. The map F is clearly continuous on $\mathbb{R}^3 \setminus \{z = 1\}$, therefore $\boldsymbol{x}$ is a homeomorphism.

(iii) The linear independence of $\partial_u \boldsymbol{x}(u,v)$ and $\partial_v \boldsymbol{x}(u,v)$ for all $(u,v) \in U$ is equivalent to

$$\partial_u \boldsymbol{x}(u,v) \times \partial_v \boldsymbol{x}(u,v) \neq 0 \quad \text{for all} \quad (u,v) \in U$$

We have

$$\begin{aligned}\partial_u \boldsymbol{x}(u,v) &= \frac{2}{(1+u^2+v^2)^2}(1-u^2+v^2, -2uv, 2u), \\ \partial_v \boldsymbol{x}(u,v) &= \frac{2}{(1+u^2+v^2)^2}(-2uv, 1+u^2-v^2, 2v),\end{aligned}$$

which implies

$$\partial_u \boldsymbol{x}(u,v) \times \partial_v \boldsymbol{x}(u,v) = \frac{4}{(1+u^2+v^2)^3}(-2u, -2v, 1-u^2-v^2) \neq 0$$

Alternatively, we have a quite simple (and abstract) argument for the linear independence of $\partial_u \boldsymbol{x}(u,v)$ and $\partial_v \boldsymbol{x}(u,v)$ for all $(u,v) \in U$, We have $\boldsymbol{F} \circ \boldsymbol{x} = \mathrm{id}_U$, hence, by the chain rule,

$$\mathrm{d}\boldsymbol{F} \circ \mathrm{d}\boldsymbol{x} = \mathrm{id}_{\mathbb{R}^2}, \quad \text{or pointwise} \quad \big(\mathrm{d}_{\boldsymbol{x}(u,v)}\boldsymbol{F}(\,\mathrm{d}_{(u,v)}\boldsymbol{x}(\boldsymbol{w}))\big) = \boldsymbol{w}$$

for all $\boldsymbol{w} \in \mathbb{R}^2$ and $(u,v) \in U$. This shows that $\mathrm{d}\boldsymbol{x}$ is injective: $\mathrm{d}_{(u,v)}\boldsymbol{x}(\boldsymbol{w}) = 0$ implies by the above equation that $0 = \big(\mathrm{d}_{\boldsymbol{x}(u,v)}\boldsymbol{F}(\,\mathrm{d}_{(u,v)}\boldsymbol{x}(\boldsymbol{w}))\big) = \boldsymbol{w}$.

Now, the image $\mathrm{d}_{(u,v)}\boldsymbol{x}(\mathbb{R}^2)$ of an injective linear map has the same dimension as the preimage, i.e., is two-dimensional. Hence, $\partial_u \boldsymbol{x}(u,v)$ and $\partial_v \boldsymbol{x}(u,v)$ are linearly independent for all $(u,v) \in U$.

6.3. Show that each of the following is a surface:

(a) a cylinder $\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}$;

(b) a two-sheet hyperboloid given by $\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 - z^2 = -1\}$.

In each case find a covering of the surface by coordinate neighborhoods and give a sketch of the surface indicating the coordinate neighbourhoods you have used.

Solution:

(a) Let $U = \{(u,v) \in \mathbb{R}^2 \,|\, |v| < 1\}$. Let us consider four charts:

$$\begin{aligned}
V_1 &= \{y > 0\}, & \boldsymbol{x}_1(u,v) &= (v, \sqrt{1-v^2}, u)\\
V_2 &= \{y < 0\}, & \boldsymbol{x}_2(u,v) &= (v, -\sqrt{1-v^2}, u)\\
V_3 &= \{x > 0\}, & \boldsymbol{x}_3(u,v) &= (\sqrt{1-v^2}, v, u)\\
V_4 &= \{x < 0\}, & \boldsymbol{x}_4(u,v) &= (-\sqrt{1-v^2}, v, u)
\end{aligned}$$

Clearly, all these maps are smooth, they are homeomorphisms onto their images (check this!), and the images cover the entire cylinder. Linear independence of partial derivatives can be verified by direct computation.

Alternatively, we could deal with two charts only. Namely, let $U = \{(u,v) \in \mathbb{R}^2 \,|\, |v| < \pi\}$, and consider two charts

$$\begin{aligned}
V_1 &= \{x > -1\}, & \boldsymbol{x}_1(u,v) &= (\cos v, \sin v, u)\\
V_2 &= \{x < 1\}, & \boldsymbol{x}_2(u,v) &= (-\cos v, -\sin v, u)
\end{aligned}$$

One can easily check that all the requirements of the definition of regular surface are satisfied.

(b) Let $U = \mathbb{R}^2$. We parametrize separately the upper half and the lower half of the hyperboloid. For the upper one, we have

$$\boldsymbol{x}(u,v) = (u, v, \sqrt{x^2 + y^2 + 1}$$

For the lower one

$$\boldsymbol{x}(u,v) = (u, v, -\sqrt{x^2 + y^2 + 1}$$

Thus, every half of the hyperboloid is a graph of a smooth function, so it is a regular surface.

6.4. For $a, b > 0$, let

$$S := \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \right\}.$$

Show that S is a surface and show that at each point $p \in S$ there are two straight lines passing through p and lying in S (i. e. S is a *doubly ruled* surface).

Solution:

S is a regular surface as a graph of a smooth function $g(x, y) = x^2/a^2 - y^2/b^2$.

Let us find the two lines in S through every point $\boldsymbol{p} \in S$. Let $\boldsymbol{\beta} : \mathbb{R} \to \mathbb{R}^3$ be a line through $\boldsymbol{p} = (x, y, z) \in S$ given by $\boldsymbol{\beta}(s) = \boldsymbol{p} + s\boldsymbol{v}$, where $\boldsymbol{v} = (u, v, w)$. Then

$$\boldsymbol{\beta}(s) = (x + su, y + sv, z + sw)$$

and $\boldsymbol{\beta}(s) \in S$ for every $s \in \mathbb{R}$ if and only if

$$z + sw = \frac{(x + su)^2}{a^2} - \frac{(y + sv)^2}{b^2},$$

which is equivalent to

$$z + sw = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) + \left(\frac{2xu}{a^2} - \frac{2yv}{b^2}\right) s + \left(\frac{u^2}{a^2} - \frac{v^2}{b^2}\right) s^2$$

Since the equality above must hold for every $s \in \mathbb{R}$, the coefficients of the polynomials in the left and right parts of the equality must coincide. Thus, we get

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \qquad w = \frac{2xu}{a^2} - \frac{2yv}{b^2}, \qquad 0 = \frac{u^2}{a^2} - \frac{v^2}{b^2}$$

The first equality holds since $\boldsymbol{p} \in S$. Solving the other two with respect to u, v, w we obtain two vectors (up to scaling)

$$\left(1, \frac{b}{a}, \frac{2x}{a^2} - \frac{2y}{ab}\right) \qquad \text{and} \qquad \left(1, -\frac{b}{a}, \frac{2x}{a^2} - \frac{2y}{ab}\right)$$

giving rise to equations of two lines.

In fact, there is an easier method to find these two lines. We can write the definition of S as

$$z = \left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right)$$

Now, for given $\boldsymbol{p} = (x_0, y_0, z_0) \in S$ denote

$$s = \frac{x_0}{a} - \frac{y_0}{b}$$

Then the intersection of the planes

$$\left\{\frac{x}{a} - \frac{y}{b} = s\right\} \qquad \text{and} \qquad \left\{\frac{x}{a} + \frac{y}{b} = \frac{z}{s}\right\}$$

is a line through $\boldsymbol{p}$ lying in S. Similarly, we can denote

$$t = \frac{x_0}{a} + \frac{y_0}{b},$$

and then the intersection of the planes

$$\left\{\frac{x}{a} - \frac{y}{b} = \frac{z}{t}\right\} \qquad \text{and} \qquad \left\{\frac{x}{a} + \frac{y}{b} = t\right\}$$

is also a line through $\boldsymbol{p}$ lying in S. One can easily check that these two lines are distinct.

6.5. $(*)$ Let S be the surface in $\mathbb{R}^3$ defined by $z = x^2 - y^2$. Show that

$$\boldsymbol{x}(u, v) = (u + \cosh v, u + \sinh v, 1 + 2u(\cosh v - \sinh v)), \qquad u, v \in \mathbb{R},$$

is a local parametrization of S. Does $\boldsymbol{x}$ parametrizes the whole surface S?

Solution:

The equality

$$1 + 2u(\cosh v - \sinh v) = (u + \cosh v)^2 - (u + \sinh v)^2$$

can be verified directly, so $\boldsymbol{x}$ is a (clearly, smooth) map from $U = \mathbb{R}^2$ to S. Note that

$$x - y = \cosh v - \sinh v = e^{-v} > 0,$$

so $\boldsymbol{x}$ parametrizes only the part $\{x > y\}$ of the surface.

To show that $\boldsymbol{x}$ is a homeomorphism between $\mathbb{R}^2$ and $S \cap \{x > y\}$ let us first check that $\boldsymbol{x}$ is injective: if it is not, then there exist two different pairs of numbers (u_1, v_1) and (u_2, v_2) such that

$$\begin{aligned}
&u_1 + \cosh v_1 = u_2 + \cosh v_2,\\
&u_1 + \sinh v_1 = u_2 + \sinh v_2,\\
&1 + 2u_1(\cosh v_1 - \sinh v_1) = 1 + 2u_2(\cosh v_2 - \sinh v_2).
\end{aligned}$$

Substracting the second equation from the first, we see that $\cosh v_1 - \sinh v_1 = \cosh v_2 - \sinh v_2$. As $\cosh v \neq \sinh v$, this (together with the third equation) imples that $u_1 = u_2$, and hence, by the first equation, $v_1 = v_2$, which contradicts to the assumption that $(u_1, v_1) \neq (u_2, v_2)$. Hence, $\boldsymbol{x}$ is injective.

Now, we compute u and v in terms of (x, y, z) (this will give us both existence of $\mathbf{x}^{-1}$ and its continuity). We already know that

$$v = -\ln(x - y)$$

Now we can compute

$$u = x - \cosh v = x - \frac{1}{2}\left(x - y + \frac{1}{x - y}\right) = \frac{1}{2}\left(x + y - \frac{1}{x - y}\right)$$

which is defined (and continuous) for every point of S with $x > y$. Therefore, (u, v) can be defined for every point $(x, y, z) \in S \cap \{x > y\}$ uniquely, the map is continuous, so $\boldsymbol{x}$ is a homeomorphism.

Linear independence of partial derivatives can be verified by a simple computation. Indeed,

$$\begin{aligned}
\partial_u \boldsymbol{x}(u, v) &= (1, 1, \cosh v - \sinh v),\\
\partial_v \boldsymbol{x}(u, v) &= (\sinh v, \cosh v, u(\sinh v - \cosh v))
\end{aligned}$$

which are linearly independent since $\sinh v$ is never equal to $\cosh v$.

6.6. Show that

(a) the cone $\{x^2 + y^2 - z^2 = 0\}$ is not a regular surface;

(b) the one-sheet cone $\{x^2 + y^2 - z^2 = 0,\ z \geq 0\}$ is not a regular surface.

Solution:

(a) Suppose that there exists a homeomorphism $\boldsymbol{x} : U \to S \cap V$ between an open set $U \in \mathbb{R}^2$ and a neighborhood of the origin in the cone (we may assume that both U and V are homeomorphic to a ball).

Let $q \in U$ such that $\boldsymbol{x}(q) = (0,0,0)$. Then $\boldsymbol{x} : U \setminus \{q\} \to (S \setminus \{(0,0,0)\} \cap V$ should also be a homeomorphism. However, $U \setminus \{q\}$ is just a punctured disc, and $S \setminus \{(0,0,0)\} \cap V$ is a disjoint union of two punctured discs. Now one can prove that there is no homeomorphism between $U \setminus \{q\}$ and $(S \setminus \{(0,0,0)\} \cap V$.

(b) The proof immediately follows from the following (easy) fact: if $S \subset \mathbb{R}^3$ is a regular surface, $\boldsymbol{p} \in S$, then there is a neighborhood V of $\boldsymbol{p}$ in $\mathbb{R}^3$ such that $S \cap V$ is the graph of a smooth function of the form $z = f(x,y)$ or $y = g(x,z)$ or $x = h(y,z)$ (this follows directly from the Implicit Function Theorem, see Proposition 3 in Section 2-2 of Do Carmo's book).