Durham University
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Differential Geometry 7.11.2019

Homework 5-6
Starred problems due on Thursday, 21 November.

Space curves - 2

5.1. (*) A curve $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is called a *(generalized) helix* if its tangent lines make a constant angle with some fixed direction in $\mathbb{R}^3$.

(a) Prove that the curve

$$\boldsymbol{\alpha}(s) = \Big(\frac{a}{c}\int_{s_0}^{s} \sin\vartheta(v)\,\mathrm{d}v, \frac{a}{c}\int_{s_0}^{s} \cos\vartheta(v)\,\mathrm{d}v, \frac{b}{c}s\Big),$$

with $s_0 \in I$, $c^2 = a^2 + b^2$, $a \neq 0$, $b \neq 0$ and $\vartheta'(s) > 0$ is a (generalized) helix.

(b) Assume that $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular curve with $\tau(s) \neq 0$ for all $s \in I$. Prove that $\boldsymbol{\alpha}$ is a (generalized) helix if and only if κ/τ is constant.

5.2. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be regular curves in $\mathbb{R}^3$ such that, for each u, the principal normals $\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$ and $\boldsymbol{n}_{\boldsymbol{\beta}}(u)$ are parallel. Prove that the angle between $\boldsymbol{t}_{\boldsymbol{\alpha}}(u)$ and $\boldsymbol{t}_{\boldsymbol{\beta}}(u)$ is independent of u. Prove also that if the line through $\boldsymbol{\alpha}(u)$ in direction $\boldsymbol{n}_{\boldsymbol{\alpha}(u)}$ coincides with the line through $\boldsymbol{\beta}(u)$ in direction $\boldsymbol{n}_{\boldsymbol{\beta}(u)}$ then

$$\boldsymbol{\beta}(u) = \boldsymbol{\alpha}(u) + r\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$$

for some real number r.

5.3. (*) Let $\boldsymbol{\alpha}$ be the curve in $\mathbb{R}^3$ given by

$$\boldsymbol{\alpha}(u) = e^u(\cos u, \sin u, 1), \qquad u \in \mathbb{R}.$$

If $0 < \lambda_0 < \lambda_1$, find the length of the segment of $\boldsymbol{\alpha}$ which lies between the planes $z = \lambda_0$ and $z = \lambda_1$. Show also that the curvature and torsion of $\boldsymbol{\alpha}$ are both inversely proportional to e^u.

5.4. Let $\boldsymbol{\alpha}$ be a curve parametrized by arc length with nowhere vanishing curvature κ and torsion τ. Show that if the trace of $\boldsymbol{\alpha}$ lies on a sphere then

$$\frac{\tau}{\kappa} = \Big(\frac{\kappa'}{\tau\kappa^2}\Big)'.$$

Is the converse true?

5.5. Let $\boldsymbol{\alpha}$ be a regular curve parametrized by arc length with $\kappa > 0$ and $\tau \neq 0$. Denote by $\boldsymbol{n}$ and $\boldsymbol{b}$ the principal normal and the binormal of $\boldsymbol{\alpha}$.

(a) If $\boldsymbol{\alpha}$ lies on a sphere with center $\boldsymbol{c} \in \mathbb{R}^3$ and radius $r > 0$, show that

$$\boldsymbol{\alpha} - \boldsymbol{c} = -\rho\boldsymbol{n} - \rho'\sigma\boldsymbol{b},$$

where $\rho = 1/\kappa$ and $\sigma = -1/\tau$. Deduce that $r^2 = \rho^2 + (\rho'\sigma)^2$.

(b) Conversely, if $\rho^2 + (\rho'\sigma)^2$ has constant value r^2 and $\rho' \neq 0$, show that $\boldsymbol{\alpha}$ lies on a sphere of radius r.

Hint: Show that the curve $\boldsymbol{\alpha} + \rho\boldsymbol{n} + \rho'\sigma\boldsymbol{b}$ is constant.

Surfaces - 1

6.1. Let $U \subset \mathbb{R}^2$ be an open set. Show that the set

$$\{(x,y,z) \in \mathbb{R}^3 \,|\, z = 0 \text{ and } (x,y) \in U\}$$

is a regular surface.

6.2. Stereographic projection
Let $S^2(1) = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1\}$ be a 2-dimensional unit sphere. For $(u,v) \in \mathbb{R}^2$, let $\boldsymbol{x}(u,v)$ be the point of intersection of the line in $\mathbb{R}^3$ through $(u,v,0)$ and $(0,0,1)$ with $S^2(1)$ (different from $(0,0,1)$).

(a) Find an explicit formula for $\boldsymbol{x}(u,v)$.

(b) Let P be the plane given by $\{z=1\}$, and for $(x,y,z) \in \mathbb{R}^3 \setminus P$, let $\boldsymbol{F}(x,y,z) \in \mathbb{R}^2$ be such that $(\boldsymbol{F}(x,y,z),0) \in \mathbb{R}^3$ is the intersection with the (x,y)-plane of the line through $(0,0,1)$ and (x,y,z). Show that

$$\boldsymbol{F}(x,y,z) = \frac{1}{1-z}(x,y).$$

(c) Show that $\boldsymbol{F} \circ \boldsymbol{x} = \mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$ and deduce that $\boldsymbol{x}$ is a local parametrization of $S^2(1) \setminus \{(0,0,1)\}$.

6.3. Show that each of the following is a surface:

(a) a cylinder $\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2 = 1\}$;

(b) a two-sheet hyperboloid given by $\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2-z^2 = -1\}$.

In each case find a covering of the surface by coordinate neighborhoods and give a sketch of the surface indicating the coordinate neighbourhoods you have used.

6.4. For $a, b > 0$, let

$$S := \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \right\}.$$

Show that S is a surface and show that at each point $p \in S$ there are two straight lines passing through p and lying in S (i. e. S is a *doubly ruled* surface).

6.5. $(*)$ Let S be the surface in $\mathbb{R}^3$ defined by $z = x^2 - y^2$. Show that

$$\boldsymbol{x}(u,v) = (u + \cosh v, u + \sinh v, 1 + 2u(\cosh v - \sinh v)), \qquad u, v \in \mathbb{R},$$

is a local parametrization of S. Does $\boldsymbol{x}$ parametrizes the whole surface S?

6.6. Show that

(a) the cone $\{x^2+y^2-z^2 = 0\}$ is not a regular surface;

(b) the one-sheet cone $\{x^2+y^2-z^2 = 0,\; z \geq 0\}$ is not a regular surface.

Hint: in (b) you need to prove that for *every* parametrization of the neighborhood of the origin the regularity condition fails.