Solutions 7-8

7.1. (∗) (a) Parametrize the hyperbolic paraboloid S from Exercise 6.4 as a ruled surface (i.e., find a curve $\boldsymbol{\alpha}(v) \subset S$ and a curve $\boldsymbol{w}(v)$ such that $\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$ will be a parametrization of S).

(b) Now let S be an arbitrary ruled surface, and let $\boldsymbol{x} : J \times I \to \mathbb{R}^3$, $\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$ be a parametrization of S such that $|\boldsymbol{w}(v)| = 1$ for all $v \in I$, where $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular space curve and I, J are intervals in $\mathbb{R}$. A curve $\boldsymbol{\beta} : I \to \mathbb{R}^3$ lying in S is called a *curve of striction* if $\boldsymbol{\beta}'(v) \cdot \boldsymbol{w}'(v) = 0$ for all $v \in I$. Find the curve of striction of the ruled surface in (a) with $a = b = 1$ (using either one of the rulings).

Solution:

(a) Take as $\boldsymbol{\alpha}$ the intersection of the paraboloid with the plane $y = 0$:

$$\boldsymbol{\alpha}(v) = (v, 0, v^2/a^2)$$

From Exercise 6.4 we know that every point $(x, y, z) \in S$ is contained in a line in the direction $(1, b/a, 2x/a^2 - 2y/ab)$, and the line itself is entirely contained in S. Taking $\boldsymbol{\alpha}(v)$ as $(x, y, z) \in S$, we see that the line through $\boldsymbol{\alpha}(v)$ has a direction vector $\boldsymbol{w}(v) = (1, b/a, 2v/a^2)$. Thus, S can be parametrized as

$$\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v) = (v, 0, v^2/a^2) + u(1, b/a, 2v/a^2) = (v + u, ub/a, (v^2 + 2uv)/a^2)$$

(b) If $a = b = 1$, we have a parametrization of the paraboloid

$$\boldsymbol{x}(u,v) = (v, 0, v^2) + u(1, 1, 2v)$$

Normalizing the direction vector computed in (a), we can write this as

$$\boldsymbol{x}(u,v) = (v, 0, v^2) + u\frac{(1, 1, 2v)}{\sqrt{2 + 4v^2}} = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v),$$

so the new (unit) direction vector $\boldsymbol{w}(v) = (1, 1, 2v)/\sqrt{2 + 4v^2}$.

Now we write

$$\boldsymbol{\beta}(v) = \boldsymbol{\alpha}(v) + u(v)\boldsymbol{w}(v),$$

so

$$\boldsymbol{\beta}'(v) = \boldsymbol{\alpha}'(v) + u'(v)\boldsymbol{w}(v) + u(v)\boldsymbol{w}'(v)$$

The assumption $\boldsymbol{\beta}'(v) \cdot \boldsymbol{w}'(v) = 0$ implies

$$0 = \boldsymbol{\beta}'(v)\cdot\boldsymbol{w}'(v) = (\boldsymbol{\alpha}'(v) + u'(v)\boldsymbol{w}(v) + u(v)\boldsymbol{w}'(v))\cdot\boldsymbol{w}'(v) = \boldsymbol{\alpha}'(v)\cdot\boldsymbol{w}'(v) + u'(v)\underbrace{\boldsymbol{w}(v)\cdot\boldsymbol{w}'(v)}_{=0} + u(v)\boldsymbol{w}'(v)\cdot\boldsymbol{w}'(v),$$

so we have

$$u(v) = -\frac{\boldsymbol{\alpha}'(v)\cdot\boldsymbol{w}'(v)}{\|\boldsymbol{w}'(v)\|^2}$$

Let us compute $\boldsymbol{w}'(v)$, and then the numerator and the denominator of the expression above.

$$\boldsymbol{w}'(v) = \left(\frac{(1,1,2v)}{\sqrt{2+4v^2}}\right)' = \frac{-4v}{(2+4v^2)^{3/2}}(1,1,2v) + \frac{(0,0,2)}{\sqrt{2+4v^2}} = -\frac{4}{(2+4v^2)^{3/2}}(v,v,-1),$$

so

$$\|\boldsymbol{w}'(v)\|^2 = \frac{8}{(2+4v^2)^2}$$

Since $\boldsymbol{\alpha}'(v) = (1,0,2v)$, we have

$$\boldsymbol{\alpha}'(v)\cdot\boldsymbol{w}'(v) = -(1,0,2v)\cdot\frac{4}{(2+4v^2)^{3/2}}(v,v,-1) = \frac{4v}{(2+4v^2)^{3/2}},$$

and

$$u(v) = -\frac{\boldsymbol{\alpha}'(v)\cdot\boldsymbol{w}'(v)}{\|\boldsymbol{w}'(v)\|^2} = -\frac{4v}{(2+4v^2)^{3/2}}\bigg/\frac{8}{(2+4v^2)^2} = -\frac{v}{2}(2+4v^2)^{1/2},$$

which implies

$$\boldsymbol{\beta}(v) = \boldsymbol{\alpha}(v) + u(v)\boldsymbol{w}(v) = (v,0,v^2) - \frac{v}{2}\sqrt{2+4v^2}\frac{(1,1,2v)}{\sqrt{2+4v^2}} = (v,0,v^2) - \frac{v}{2}(1,1,2v) = \frac{v}{2}(1,-1,0)$$

One can note that $\boldsymbol{\beta}(v)$ is one of the lines from the second family of lines forming S.

7.2. (a) Show that the set S of $(x,y,z)\in\mathbb{R}^3$ fulfilling the equation $xz+y^2=1$ is a surface.

(b) Let $\boldsymbol{\alpha},\boldsymbol{w}:\mathbb{R}\to\mathbb{R}^3$ be given by

$$\boldsymbol{\alpha}(v) = (\cos v, \sin v, \cos v) \quad\text{and}\quad \boldsymbol{w}(v) = (1+\sin v, -\cos v, -1+\sin v).$$

Show that for all $v\in\mathbb{R}$ there are two straight lines through $\boldsymbol{\alpha}(v)$, one of which is in direction $\boldsymbol{w}(v)$, both of which lie on S. If $\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$, $u\in\mathbb{R}$, $0<v<2\pi$, show that $\boldsymbol{x}$ is a local parametrization of S.

Solution:

(a) Computing the gradient of a smooth function $f(x,y,z) = xz+y^2$ we see that

$$\nabla f(x,y,z) = (z,2y,x)$$

is equal to zero if and only $(x,y,z) = (0,0,0)$, which implies that 1 is a regular value of f, so S is a regular surface.

(b) This can be solved similar to Exercise 6.4. We want to find a line in S through every point $\boldsymbol{\alpha}(v)$, i.e. a vector $\boldsymbol{w}(v) = (a(v),b(v),c(v))$ such that the line $\boldsymbol{\beta}_v(u) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$ lies in S. Then

$$\boldsymbol{\beta}_v(u) = (ua+\cos v, ub+\sin v, uc+\cos v)$$

and $\boldsymbol{\beta}_v(u)\in S$ for every $u\in\mathbb{R}$ if and only if

$$(ua+\cos v)(uc+\cos v) + (ub+\sin v)^2 = 1,$$

which is equivalent to

$$u^2(ac+b^2) + u((a+c)\cos v + 2b\sin v) + 1 = 1$$

for every $u\in\mathbb{R}$, which implies

$$a(v)c(v) + b^2(v) = (a(v)+c(v))\cos v + 2b(v)\sin v = 0$$

The equality $(a(v) + c(v)) \cos v + 2b(v) \sin v = 0$ implies, up to scaling, that $a(v) + c(v) = 2 \sin v$ and $b(v) = -\cos v$. Together with $a(v)c(v) + b^2(v) = 0$ this leads to

$$\boldsymbol{w}(v) = (a(v), b(v), c(v)) = (\pm 1 + \sin v, -\cos v, \mp 1 + \sin v)$$

As in Exercise 6.4, there is an easier way to proceed. Changing coordinates (orthogonally) by $x = (x' - z')/2$ and $z = (x' + z')/2$ we get an equation of a one-sheeted hyperboloid, for which we know that it is doubly ruled.

7.3. Determine all surfaces of revolution which are also ruled surfaces.

Solution:

Let S be such a surface. Since S is a surface of revolution, it contains a circle $\{r(\cos u, \sin u, 0)\}$. Since S is a ruled surface, it contains a line through $\boldsymbol{p} = (r, 0, 0) \in S$ in the direction $\boldsymbol{w} = (a, b, c)$ (we assume that S contains the entire line, otherwise we just get a piece of this surface). Then the whole S is obtained by rotation of the line around z-axis. Therefore, the surface is completely defined by $r > 0$ and a direction (a, b, c). The parameter r does not change the type of S and is responsible for "scaling" only. Let us look how does S depend on (a, b, c).

If the vector (a, b, c) lies in xy-plane (i.e., $c = 0$), then S is not a surface of revolution (since there is no regular curve $\boldsymbol{\alpha}(v)$ in xz-plane). Thus, $c \neq 0$, and we may assume without loss of generality that $\boldsymbol{w} = (a, b, 1)$.

If $a = b = 0$, we get a cylinder

$$x^2 + y^2 = r^2$$

If $a \neq 0, b = 0$, then the line meets z-axis at the point $(0, 0, -r/a)$. Rotating this line around z-axis, we obtain a cone with equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \left(z + \frac{r}{a}\right)^2 = 0$$

(check this!)

If $b \neq 0$, then the line does not meet z-axis, and one can easily see that we get a one-sheeted hyperboloid (shifted along z-axis). Since the hyperbolid is obtained by rotation around z-axis, it should have an equation

$$\frac{x^2}{c^2} + \frac{y^2}{c^2} - (z - d)^2 = k^2$$

for some real numbers c, d and k (check this!). Now, proceeding as in Exercise 7.2(b), we compute an equation to be

$$\frac{x^2}{a^2 + b^2} + \frac{y^2}{a^2 + b^2} - \left(z + \frac{ra}{a^2 + b^2}\right)^2 = \frac{r^2 b^2}{(a^2 + b^2)^2}$$

One can easily check that the line through $(r, 0, 0)$ in the direction $(a, b, 1)$ is contained in S, and thus every rotation of it as well (since the equation is invariant with respect to rotation around z-axis, i.e. with respect to substitution (x, y, z) by $(x \cos u, y \sin u, z)$).

7.4. $(*)$ Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = (x + y + z - 1)^2$.

(a) Find the points at which $\operatorname{grad} f = 0$.

(b) For which values of c the level set $S := \{p = (x, y, z) \in \mathbb{R}^3 \mid f(p) = c\}$ is a surface?

(c) What is the level set $f(p) = c$?

(d) Repeat (a) and (b) using the function $f(x, y, z) = xyz^2$.

Solution:

(a) $\nabla f(x, y, z) = 2(x + y + z - 1)(1, 1, 1)$, which implies that $\nabla f = 0$ if and only if $x + y + z = 1$.

(b) According to (a), $\nabla f = 0$ if and only if $x + y + z - 1 = 0$, which is equivalent to $f(x, y, z) = 0$. Thus, the only singular value of f is 0, and for any $c \neq 0$ the level set $f(p) = c$ is a regular surface.

However, although $c = 0$ is a singular value of f, for $c = 0$ the level set $f(p) = c$ is also a regular surface: $f(p) = 0$ is a plane $x + y + z = 1$ which is clearly regular.

(c) The equation $(x + y + z - 1)^2 = c$ is equivalent to $(x + y + z - 1) = \pm\sqrt{c}$, so it is a union of two parallel planes for $c \neq 0$, one plane for $c = 0$ and empty set for $c < 0$.

(d) $\nabla f(x, y, z) = (yz^2, xz^2, 2xyz) = z(yz, xz, 2xy)$, which implies that $\nabla f = 0$ if and only if $z = 0$ or $x = y = 0$, so the only singular value of f is 0, and for any $c \neq 0$ the level set $f(p) = c$ is a regular surface. The level set $f(p) = 0$ is a union of three coordinate planes, so it is not a regular surface (the "bad" points are ones lying on coordinate axes, check this!)

7.5. Möbius band

Let S be the image of the function $f : \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3, (\varepsilon > 0)$, defined by

$$f(u, v) = \left(\left(2 - v \sin \frac{u}{2}\right) \sin u,\ \left(2 - v \sin \frac{u}{2}\right) \cos u,\ v \cos \frac{u}{2}\right).$$

Show that, for ε sufficiently small, S is a surface in $\mathbb{R}^3$ which may be covered by two coordinate neighborhoods. Give a sketch of the surface indicating the curves $u = \text{const}$ and $v = \text{const}$ (such curves are called *coordinate curves*).

Solution:

(a) Let us write $f(u, v)$ as

$$f(u, v) = \underbrace{(2 \sin u, 2 \cos u, 0)}_{=:\boldsymbol{\alpha}(u)} + v \underbrace{\left(-\sin \frac{u}{2} \sin u,\ -\sin \frac{u}{2} \cos u,\ \cos \frac{u}{2}\right)}_{=:\boldsymbol{w}(u)}$$

By the form of f, S is a ruled surface, and one can easily see that $\boldsymbol{\alpha}(u)$ is regular, and $\boldsymbol{\alpha}'(u)$ and $\boldsymbol{w}(u)$ are not collinear for all $u \in \mathbb{R}$. Now, to have a regular surface, we need the intervals through different points of $\boldsymbol{\alpha}(u)$ to be disjoint. A straightforward calculation shows that this holds for small ε (say, for $0 < \varepsilon < 2$).

In fact, the latter can be shown geometrically. One can note that the line $\boldsymbol{l}_u(v) = f(u, v)$ through $\boldsymbol{\alpha}(u)$ in the direction $\boldsymbol{w}(u)$ meets the z-axis at the point $(0, 0, \cot u/2)$ (unless $u = 0$: in this case $\boldsymbol{l}_u(v)$ is parallel to z-axis). Therefore, if two such lines $\boldsymbol{l}_{u_1}(v)$ and $\boldsymbol{l}_{u_2}(v)$ intersect, they should be contained in a plane passing through the z-axis, and thus intersect the circle $\{x^2 + y^2 = 4,\ z = 0\}$ (which is the trace of $\boldsymbol{\alpha}$) in two opposite points only, which is clearly not the case (unless $u_2 = u_1 + n\pi$) since the lines meet $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_1)$ and $\boldsymbol{\alpha}(u_2)$. The condition that the intervals lying on lines $\boldsymbol{l}_u(v)$ and $\boldsymbol{l}_{u+\pi}(v)$ do not intersect is guaranteed by the assumption $\varepsilon < 2$.

(b) Clearly, f is not injective: $\boldsymbol{\alpha}$ has a period 2π, so $f(u_0 + 2\pi, 0) = f(u_0, 0)$. However, if we take an open set $U_1 = (0, 2\pi) \times (-\varepsilon, \varepsilon)$, then the restriction of f on U_1 is injective, and the image of U_1 is the whole Möbius strip except one interval $f(0 \times (-\varepsilon, \varepsilon))$. Taking $U_2 = (-\pi, \pi) \times (-\varepsilon, \varepsilon)$, we see that $f(U_1) \cup f(U_2) = S$.

7.6. Real projective plane (bonus problem)

Let $f : \mathbb{R}^3 \to \mathbb{R}^5$ be defined by

$$f(x,y,z) = \Big(yz, zx, xy, \frac{1}{2}(x^2 - y^2), \frac{1}{2\sqrt{3}}(x^2 + y^2 - 2z^2)\Big).$$

Show that:

(a) $f(x,y,z) = f(x',y',z')$ if and only if $(x,y,z) = \pm(x',y',z')$;

(b) the image $S = f(S^2(1))$ of the unit sphere $S^2(1)$ in $\mathbb{R}^3$ is a surface in $\mathbb{R}^5$.

The surface S is often written as $\mathbb{R}P^2$ and is called the *real projective plane.* Note that it can be identified with the set of lines through the origin in $\mathbb{R}^3$.

Hint: you may find it helpful to consider the open subsets W_x, W_y, W_z of $\mathbb{R}^5$ given by

$$\begin{aligned} W_x &= \Big\{(x_1, x_2, x_3, x_4, x_5) \,|\, x_4 + \tfrac{1}{\sqrt{3}} x_5 + \tfrac{1}{3} > 0\Big\}, \\ W_y &= \Big\{(x_1, x_2, x_3, x_4, x_5) \,|\, -x_4 + \tfrac{1}{\sqrt{3}} x_5 + \tfrac{1}{3} > 0\Big\}, \\ W_z &= \Big\{(x_1, x_2, x_3, x_4, x_5) \,|\, x_5 < \tfrac{1}{2\sqrt{3}}\Big\}, \end{aligned}$$

and use the fact that the intersections of S with W_x W_y and W_z are the images of the hemispheres of $S^2(1)$ given by $x > 0, y > 0$ and $z > 0$, respectively.

8.1. (a) Let $\boldsymbol{x} : U \to S$ be a local parametrization of a surface S in some neighborhood of a point $\boldsymbol{p} = (x_0, y_0, z_0) \in S$. Show that the tangent plane to S at $\boldsymbol{p}$ has an equation

$$\left(\frac{\partial \boldsymbol{x}}{\partial u}(\boldsymbol{p}) \times \frac{\partial \boldsymbol{x}}{\partial v}(\boldsymbol{p})\right) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, and let $c \in f(\mathbb{R}^3)$ be a regular value of f. Show that the tangent plane of a regular surface

$$S = \{(x,y,z) \,|\, f(x,y,z) = c\}$$

at the point $\boldsymbol{p} = (x_0, y_0, z_0) \in S$ has equation

$$\frac{\partial f}{\partial x}(\boldsymbol{p})(x - x_0) + \frac{\partial f}{\partial y}(\boldsymbol{p})(y - y_0) + \frac{\partial f}{\partial z}(\boldsymbol{p})(z - z_0) = 0$$

Solution:

(a) By definition, the tangent plane to S at $\boldsymbol{p} \in S$ is spanned by vectors $\frac{\partial \boldsymbol{x}}{\partial u}(\boldsymbol{p})$ and $\frac{\partial \boldsymbol{x}}{\partial v}(\boldsymbol{p})$ and passes through $\boldsymbol{p}$. Let Π be the plane defined by the equation above. Since $\frac{\partial \boldsymbol{x}}{\partial u}(\boldsymbol{p}) \times \frac{\partial \boldsymbol{x}}{\partial v}(\boldsymbol{p})$ is orthogonal to both $\frac{\partial \boldsymbol{x}}{\partial u}(\boldsymbol{p})$ and $\frac{\partial \boldsymbol{x}}{\partial v}(\boldsymbol{p})$, the both partial derivatives lie in Π. Now, the point $\boldsymbol{p} = (x_0, y_0, z_0)$ itself clearly satisfies the equation.

(b) If $\boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to S$ is any curve with $\boldsymbol{\alpha}(0) = \boldsymbol{p}$, then $f(\boldsymbol{\alpha}(u)) \equiv c$. Differentiating, we obtain

$$\nabla f(p) \cdot \boldsymbol{\alpha}'(0) = 0,$$

which implies that the tangent plane is orthogonal to the gradient $\nabla f(p) = \left(\frac{\partial f}{\partial x}(\boldsymbol{p}), \frac{\partial f}{\partial y}(\boldsymbol{p}), \frac{\partial f}{\partial z}(\boldsymbol{p})\right)$.

8.2. ($*$) Show that the tangent plane of one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ at point $(x, y, 0)$ is parallel to the z-axis.

Solution:

Using Exercise 8.1(b), we see that the tangent plane at point $(x_0, y_0, 0)$ of the hyperboloid has an equation

$$x_0(x - x_0) + y_0(y - y_0) = 0$$

which is clearly parallel to z-axis.

8.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Define a surface S as

$$S = \{(x, y, z) \mid xf(y/x) - z = 0,\ x \neq 0\}$$

Show that all tangent planes of S pass through the origin $(0, 0, 0)$.

Solution:

The surface is the graph of a smooth function $z = xf(y/x)$, so it has a parametrization

$$\boldsymbol{x}(x, y) = (x, y, xf(y/x))$$

First, we compute $\frac{\partial \boldsymbol{x}}{\partial x}$ and $\frac{\partial \boldsymbol{x}}{\partial y}$, and then use Exercise 8.1(a).

$$\begin{aligned}
\frac{\partial \boldsymbol{x}}{\partial x}(x, y) &= \left(1, 0, f\left(\frac{y}{x}\right) - \frac{y}{x} f'\left(\frac{y}{x}\right)\right), \\
\frac{\partial \boldsymbol{x}}{\partial y}(x, y) &= \left(0, 1, f'\left(\frac{y}{x}\right)\right)
\end{aligned}$$

Thus,

$$\frac{\partial \boldsymbol{x}}{\partial x} \times \frac{\partial \boldsymbol{x}}{\partial y}(x, y) = \left(-f\left(\frac{y}{x}\right) + \frac{y}{x} f'\left(\frac{y}{x}\right), -f'\left(\frac{y}{x}\right), 1\right),$$

and an equation of the tangent plane at $(x_0, y_0, z_0) \in S$ is

$$\left(-f\left(\frac{y_0}{x_0}\right) + \frac{y_0}{x_0} f'\left(\frac{y_0}{x_0}\right), -f'\left(\frac{y_0}{x_0}\right), 1\right) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

This plane passes through the origin if and only if

$$\left(-f\left(\frac{y_0}{x_0}\right) + \frac{y_0}{x_0} f'\left(\frac{y_0}{x_0}\right), -f'\left(\frac{y_0}{x_0}\right), 1\right) \cdot (x_0, y_0, z_0) = 0$$

Indeed, taking into account that

$$f\left(\frac{y_0}{x_0}\right) = \frac{z_0}{x_0},$$

we have

$$\left(-f\left(\frac{y_0}{x_0}\right) + \frac{y_0}{x_0} f'\left(\frac{y_0}{x_0}\right), -f'\left(\frac{y_0}{x_0}\right), 1\right) \cdot (x_0, y_0, z_0) = -\frac{z_0}{x_0} x_0 + y_0 f'\left(\frac{y_0}{x_0}\right) - y_0 f'\left(\frac{y_0}{x_0}\right) + z_0 = 0$$

8.4. Let $U \subset \mathbb{R}^2$ be open, and let S_1 and S_2 be two regular surfaces with parametrizations $\boldsymbol{x} : U \to S_1$ and $\boldsymbol{y} : U \to S_2$. Define a map $\boldsymbol{\varphi} = \boldsymbol{y} \circ \boldsymbol{x}^{-1} : S_1 \to S_2$. Let $\boldsymbol{p} \in S_1$, $\boldsymbol{w} \in T_{\boldsymbol{p}}S_1$, and let $\boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to S_1$ be an arbitrary regular curve in S_1 such that $\boldsymbol{p} = \boldsymbol{\alpha}(0)$ and $\boldsymbol{\alpha}'(0) = \boldsymbol{w}$. Define $\boldsymbol{\beta} : (-\varepsilon, \varepsilon) \to S_2$ as $\boldsymbol{\beta} = \boldsymbol{\varphi} \circ \boldsymbol{\alpha}$.

(a) Show that $\boldsymbol{\beta}'(0)$ does not depend on the choice of $\boldsymbol{\alpha}$.

(b) Show that the map $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi} : T_{\boldsymbol{p}}S_1 \to T_{\boldsymbol{\varphi}(\boldsymbol{p})}S_2$ defined by $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi}(\boldsymbol{w}) = \boldsymbol{\beta}'(0)$ is linear.

Solution:

(a) Define a curve $\boldsymbol{\gamma} : (-\varepsilon, \varepsilon) \to U$ by $\boldsymbol{\alpha} = \boldsymbol{x} \circ \boldsymbol{\gamma}$, and define $\boldsymbol{q} \in U$ by $\boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{p}$. Then, by the chain rule,

$$\boldsymbol{w} = \boldsymbol{\alpha}'(0) = (\boldsymbol{x} \circ \boldsymbol{\gamma})'(0) = \mathrm{d}_{\boldsymbol{\gamma}(0)}\boldsymbol{x}(\gamma'(0)) = \mathrm{d}_{\boldsymbol{q}}\boldsymbol{x}(\gamma'(0))$$

Thus,

$$\boldsymbol{\gamma}'(0) = (\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1}(\boldsymbol{w}),$$

where by $(\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1}$ we mean the *left* inverse of $\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x}$, namely, a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$ satisfying $(\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1} \circ \mathrm{d}_{\boldsymbol{q}}\boldsymbol{x} = \mathrm{id}_{\mathbb{R}^2}$ (notice that $\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x}$ has no inverse since it is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$). In particular, we see that $\boldsymbol{\gamma}'(0)$ does not depend on the choice of $\boldsymbol{\alpha}$ but on the vector $\boldsymbol{w}$ only.

Now, we can write

$$\boldsymbol{\beta} = \boldsymbol{y} \circ \boldsymbol{\gamma},$$

and differentiating this we get

$$\boldsymbol{\beta}'(0) = (\boldsymbol{y} \circ \boldsymbol{\gamma})'(0) = \mathrm{d}_{\boldsymbol{\gamma}(0)}\boldsymbol{y}(\gamma'(0)) = \mathrm{d}_{\boldsymbol{q}}\boldsymbol{y}(\gamma'(0))$$

Therefore, $\boldsymbol{\beta}'(0)$ is completely defined by $\mathrm{d}_{\boldsymbol{q}}\boldsymbol{y}$ and $\boldsymbol{\gamma}'(0)$ which do not depend on the choice of $\boldsymbol{\alpha}$.

(b) As we have seen in (a),

$$\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi}(\boldsymbol{w}) = \boldsymbol{\beta}'(0) = \mathrm{d}_{\boldsymbol{q}}\boldsymbol{y}(\gamma'(0)) = \mathrm{d}_{\boldsymbol{q}}\boldsymbol{y}((\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1}(w)) = (\mathrm{d}_{\boldsymbol{q}}\boldsymbol{y} \circ (\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1})(\boldsymbol{w}),$$

which implies

$$\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi} = \mathrm{d}_{\boldsymbol{q}}\boldsymbol{y} \circ (\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1}$$

which is clearly linear as a composition of two linear maps.

8.5. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a regular curve with nonzero curvature parametrized by arc length. Recall that a *canal surface* (or *tubular surface*) S is a surface parametrized by

$$\boldsymbol{x}(u, v) = \boldsymbol{\alpha}(u) + r(\boldsymbol{n}(u)\cos v + \boldsymbol{b}(u)\sin v),$$

where $\boldsymbol{n}$ and $\boldsymbol{b}$ are unit normal and binormal vectors, and $r > 0$ is a sufficiently small constant. Find the equation of the tangent plane to S at $\boldsymbol{x}(u, v)$. In particular, show that the tangent plane at $\boldsymbol{x}(u, v)$ is parallel to $\boldsymbol{\alpha}'(u)$.

Solution: We use Exercise 8.1(a) to compute an equation of the tangent plane.

$$\begin{aligned}\frac{\partial \boldsymbol{x}}{\partial u}(u, v) = \boldsymbol{\alpha}'(u) + r(\boldsymbol{n}'(u)\cos v + \boldsymbol{b}'(u)\sin v) &= \boldsymbol{t} + r(-\kappa \boldsymbol{t} - \tau \boldsymbol{b})\cos v + r\tau \boldsymbol{n}\sin v = \\ &= \boldsymbol{t}(1 - r\kappa\cos v) + \boldsymbol{n}(r\tau\sin v) + \boldsymbol{b}(-r\tau\cos v),\end{aligned}$$

and

$$\frac{\partial \boldsymbol{x}}{\partial v}(u, v) = r(\boldsymbol{n}(u)(-\sin v) + \boldsymbol{b}(u)\cos v) = \boldsymbol{n}(-r\sin v) + \boldsymbol{b}(r\cos v)$$

Now, computing the cross-product, we get

$$\left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right)(u,v) = -r(1 - r\kappa\cos v)(\boldsymbol{n}(u)\cos v + \boldsymbol{b}(u)\sin v)$$

An equation of the tangent plane to S at point $\boldsymbol{x}(u_0, v_0)$ with respect to variable $\boldsymbol{q} \in \mathbb{R}^3$ can be written as

$$(\boldsymbol{n}(u_0)\cos v_0 + \boldsymbol{b}(u_0)\sin v_0) \cdot (\boldsymbol{q} - (\boldsymbol{\alpha}(u_0) + r(\boldsymbol{n}(u_0)\cos v_0 + \boldsymbol{b}(u_0)\sin v_0)) = 0$$

Since $\boldsymbol{n}(u_0)\cos v_0 + \boldsymbol{b}(u_0)\sin v_0$ is a unit vector, this is equivalent to

$$(\boldsymbol{n}(u_0)\cos v_0 + \boldsymbol{b}(u_0)\sin v_0) \cdot (\boldsymbol{q} - \boldsymbol{\alpha}(u_0)) = r$$

In particular, the vector $\boldsymbol{n}(u_0)\cos v_0 + \boldsymbol{b}(u_0)\sin v_0$ is orthogonal to $\boldsymbol{t}(u_0)$ as a linear combination of $\boldsymbol{n}(u_0)$ and $\boldsymbol{b}(u_0)$, so the plane is parallel to $\boldsymbol{t}(u_0)$.