Durham University
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Homework 7-8
Starred problems due on Tuesday, 3 December.

Surfaces - 2

7.1. ($*$) (a) Parametrize the hyperbolic paraboloid S from Exercise 6.4 as a ruled surface (i.e., find a curve $\boldsymbol{\alpha}(v) \subset S$ and a curve $\boldsymbol{w}(v)$ such that $\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$ will be a parametrization of S).

(b) Now let S be an arbitrary ruled surface, and let $\boldsymbol{x} : J \times I \to \mathbb{R}^3$, $\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$ be a parametrization of S such that $|\boldsymbol{w}(v)| = 1$ for all $v \in I$, where $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular space curve and I, J are intervals in $\mathbb{R}$. A curve $\boldsymbol{\beta} : I \to \mathbb{R}^3$ lying in S is called a *curve of striction* if $\boldsymbol{\beta}'(v) \cdot \boldsymbol{w}'(v) = 0$ for all $v \in I$. Find the curve of striction of the ruled surface in (a) with $a = b = 1$ (using either one of the rulings).

Hint: You may assume $\boldsymbol{\beta}(v) = \boldsymbol{\alpha}(v) + u(v)\boldsymbol{w}(v)$.

7.2. (a) Show that the set S of $(x,y,z) \in \mathbb{R}^3$ fulfilling the equation $xz + y^2 = 1$ is a surface.

(b) Let $\boldsymbol{\alpha}, \boldsymbol{w} : \mathbb{R} \to \mathbb{R}^3$ be given by

$$\boldsymbol{\alpha}(v) = (\cos v, \sin v, \cos v) \quad \text{and} \quad \boldsymbol{w}(v) = (1 + \sin v, -\cos v, -1 + \sin v).$$

Show that for all $v \in \mathbb{R}$ there are two straight lines through $\boldsymbol{\alpha}(v)$, one of which is in direction $\boldsymbol{w}(v)$, both of which lie on S. If $\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$, $u \in \mathbb{R}$, $0 < v < 2\pi$, show that $\boldsymbol{x}$ is a local parametrization of S.

7.3. Determine all surfaces of revolution which are also ruled surfaces.

7.4. ($*$) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = (x + y + z - 1)^2$.

(a) Find the points at which $\operatorname{grad} f = 0$.

(b) For which values of c the level set $S := \{p = (x,y,z) \in \mathbb{R}^3 \mid f(p) = c\}$ is a surface?

(c) What is the level set $f(p) = c$?

(d) Repeat (a) and (b) using the function $f(x,y,z) = xyz^2$.

7.5. Möbius band
Let S be the image of the function $f : \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3, (\varepsilon > 0)$, defined by

$$f(u,v) = \left(\left(2 - v\sin\frac{u}{2}\right)\sin u,\ \left(2 - v\sin\frac{u}{2}\right)\cos u,\ v\cos\frac{u}{2}\right).$$

Show that, for ε sufficiently small, S is a surface in $\mathbb{R}^3$ which may be covered by two coordinate neighborhoods. Give a sketch of the surface indicating the curves $u = \text{const}$ and $v = \text{const}$ (such curves are called *coordinate curves*).

7.6. Real projective plane (bonus problem)
Let $f : \mathbb{R}^3 \to \mathbb{R}^5$ be defined by

$$f(x,y,z) = \left(yz, zx, xy, \frac{1}{2}(x^2 - y^2), \frac{1}{2\sqrt{3}}(x^2 + y^2 - 2z^2)\right).$$

Show that:

(a) $f(x,y,z) = f(x',y',z')$ if and only if $(x,y,z) = \pm(x',y',z')$;

(b) the image $S = f(S^2(1))$ of the unit sphere $S^2(1)$ in $\mathbb{R}^3$ is a surface in $\mathbb{R}^5$.

The surface S is often written as $\mathbb{R}P^2$ and is called the *real projective plane*. Note that it can be identified with the set of lines through the origin in $\mathbb{R}^3$.

Tangent plane

8.1. (a) Let $\boldsymbol{x} : U \to S$ be a local parametrization of a surface S in some neiborhood of a point $\boldsymbol{p} = (x_0, y_0, z_0) \in S$. Show that the tangent plane to S at $\boldsymbol{p}$ has equation

$$\left(\frac{\partial \boldsymbol{x}}{\partial u}(\boldsymbol{p}) \times \frac{\partial \boldsymbol{x}}{\partial v}(\boldsymbol{p})\right) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, and let $c \in f(\mathbb{R}^3)$ be a regular value of f. Show that the tangent plane of the regular surface

$$S = \{(x, y, z) \mid f(x, y, z) = c\}$$

at the point $\boldsymbol{p} = (x_0, y_0, z_0) \in S$ has equation

$$\frac{\partial f}{\partial x}(\boldsymbol{p})(x - x_0) + \frac{\partial f}{\partial y}(\boldsymbol{p})(y - y_0) + \frac{\partial f}{\partial z}(\boldsymbol{p})(z - z_0) = 0$$

8.2. $(*)$ Show that the tangent plane of one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ at point $(x, y, 0)$ is parallel to the z-axis.

8.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Define a surface S as

$$S = \{(x, y, z) \mid x f(y/x) - z = 0,\ x \neq 0\}$$

Show that all tangent planes of S pass through the origin $(0, 0, 0)$.

8.4. Let $U \subset \mathbb{R}^2$ be open, and let S_1 and S_2 be two regular surfaces with parametrizations $\boldsymbol{x} : U \to S_1$ and $\boldsymbol{y} : U \to S_2$. Define a map $\boldsymbol{\varphi} = \boldsymbol{y} \circ \boldsymbol{x}^{-1} : S_1 \to S_2$. Let $\boldsymbol{p} \in S_1$, $\boldsymbol{w} \in T_{\boldsymbol{p}} S_1$, and let $\boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to S_1$ be an arbitrary regular curve in S_1 such that $\boldsymbol{p} = \boldsymbol{\alpha}(0)$ and $\boldsymbol{\alpha}'(0) = \boldsymbol{w}$. Define $\boldsymbol{\beta} : (-\varepsilon, \varepsilon) \to S_2$ as $\boldsymbol{\beta} = \boldsymbol{\varphi} \circ \boldsymbol{\alpha}$.

(a) Show that $\boldsymbol{\beta}'(0)$ does not depend on the choice of $\boldsymbol{\alpha}$.

(b) Show that the map $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi} : T_{\boldsymbol{p}} S_1 \to T_{\boldsymbol{\varphi}(\boldsymbol{p})} S_2$ defined by $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi}(\boldsymbol{w}) = \boldsymbol{\beta}'(0)$ is linear.

8.5. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a regular curve with nonzero curvature parametrized by arc length. Recall that a *canal surface* (or *tubular surface*) S is a surface parametrized by

$$\boldsymbol{x}(u, v) = \boldsymbol{\alpha}(u) + r(\boldsymbol{n}(u) \cos v + \boldsymbol{b}(u) \sin v),$$

where $\boldsymbol{n}$ and $\boldsymbol{b}$ are unit normal and binormal vectors, and $r > 0$ is a sufficiently small constant. Find the equation of the tangent plane to S at $\boldsymbol{x}(u, v)$. In particular, show that the tangent plane at $\boldsymbol{x}(u, v)$ is parallel to $\boldsymbol{\alpha}'(u)$.