Durham University
Anna Felikson

Differential Geometry 18.12.2019

Solutions 9-10

9.1. Find the coefficients of the first fundamental forms of:

(a) the *catenoid* parametrized by

$$\boldsymbol{x}(u,v) = (\cosh v \cos u, \cosh v \sin u, v), \qquad (u,v) \in U := (0,2\pi) \times \mathbb{R};$$

(b) the *helicoid* parametrized by

$$\widetilde{\boldsymbol{x}}(u,v) = (-\sinh v \sin u, \sinh v \cos u, -u), \qquad (u,v) \in U;$$

(c) the surface S_ϑ (for some $\vartheta \in \mathbb{R}$) parametrized by

$$\boldsymbol{y}_\vartheta(u,v) = (\cos\vartheta)\boldsymbol{x}(u,v) + (\sin\vartheta)\widetilde{\boldsymbol{x}}(u,v), \qquad (u,v) \in U.$$

Solution:

(a) We have

$$\begin{array}{rcl} \partial_u \boldsymbol{x}(u,v) &=& (-\cosh v \sin u, \cosh v \cos u, 0), \\ \partial_v \boldsymbol{x}(u,v) &=& (\sinh v \cos u, \sinh v \sin u, 1). \end{array}$$

This implies that

$$\begin{array}{rcl} E(u,v) &=& (-\cosh)^2 v \sin^2 u + \cosh^2 v \cos^2 u = \cosh^2 v, \\ F(u,v) &=& 0, \\ G(u,v) &=& \sinh^2 v \cos^2 u + \sinh^2 v \sin^2 u + 1 = \sinh^2 v + 1 = \cosh^2 v, \end{array}$$

i.e., the first fundamental form at $\boldsymbol{x}(u,v)$ is just a multiple of the standard inner product in $\mathbb{R}^2$ by the factor $\cosh^2 v$.

(b) We have

$$\begin{array}{rcl} \partial_u \widetilde{\boldsymbol{x}}(u,v) &=& (-\sinh v \cos u, -\sinh v \sin u, -1), \\ \partial_v \widetilde{\boldsymbol{x}}(u,v) &=& (-\cosh v \sin u, \cosh v \cos u, 0). \end{array}$$

This implies that

$$\begin{array}{rcl} \widetilde{E}(u,v) &=& (-\sinh)^2 v \cos^2 u + (-\sinh)^2 v \sin^2 u + (-1)^2 = \cosh^2 v, \\ \widetilde{F}(u,v) &=& 0, \\ \widetilde{G}(u,v) &=& (-\cosh)^2 v \sin^2 u + \cosh^2 v \cos^2 u = \cosh^2 v, \end{array}$$

i.e., the first fundamental form at $\widetilde{\boldsymbol{x}}(u,v)$ is again just a multiple of the standard inner product in $\mathbb{R}^2$ by the factor $\cosh^2 v$.

(c) Now we choose

$$\boldsymbol{y}_\vartheta(u,v) = \cos\vartheta\, \boldsymbol{x}(u,v) + \sin\vartheta\, \widetilde{\boldsymbol{x}}(u,v).$$

We obviously have

$$\begin{aligned}\partial_u \boldsymbol{y}_\vartheta &= \cos\vartheta \partial_u \boldsymbol{x} + \sin\vartheta \partial_u \widetilde{\boldsymbol{x}},\\ \partial_v \boldsymbol{y}_\vartheta &= \cos\vartheta \partial_v \boldsymbol{x} + \sin\vartheta \partial_v \widetilde{\boldsymbol{x}}.\end{aligned}$$

We easily check that $\langle \partial_u \boldsymbol{x}, \partial_u \widetilde{\boldsymbol{x}}\rangle = 0 = \langle \partial_v \boldsymbol{x}, \widetilde{\partial}_v \boldsymbol{x}\rangle$ and

$$\langle \partial_u \boldsymbol{x}, \widetilde{\partial}_v \boldsymbol{x}\rangle + \langle \partial_v \boldsymbol{x}, \widetilde{\partial}_u \boldsymbol{x}\rangle = \cosh^2 v - (\sinh^2 v + 1) = 0.$$

This implies that

$$\begin{aligned}\langle \partial_u \boldsymbol{y}_\vartheta, \partial_u \boldsymbol{y}_\vartheta\rangle &= \cos^2\vartheta E + \sin^2\vartheta \widetilde{E} + 2\sin\vartheta\cos\vartheta \langle \partial_u \boldsymbol{x}, \widetilde{\partial}_u \boldsymbol{x}\rangle = \cosh^2 v,\\ \langle \partial_u \boldsymbol{y}_\vartheta, \partial_v \boldsymbol{y}_\vartheta\rangle &= \cos^2\vartheta F + \sin^2\vartheta \widetilde{F} + \sin\vartheta\cos\vartheta(\langle \partial_u \boldsymbol{x}, \widetilde{\partial}_v \boldsymbol{x}\rangle + \langle \partial_v \boldsymbol{x}, \widetilde{\partial}_u \boldsymbol{x}\rangle)\\ &= \cos^2\vartheta \cdot 0 + \sin^2\vartheta \cdot 0 + \sin\vartheta\cos\vartheta \cdot 0 = 0,\\ \langle \partial_v \boldsymbol{y}_\vartheta, \partial_v \boldsymbol{y}_\vartheta\rangle &= \cos^2\vartheta G + \sin^2\vartheta \widetilde{G} + 2\sin\vartheta\cos\vartheta \langle \partial_v \boldsymbol{x}, \partial_v \widetilde{\boldsymbol{x}}\rangle = \cosh^2 v,\end{aligned}$$

i.e., the first fundamental form at $\boldsymbol{y}_\vartheta(u,v)$ is again just a multiple of the standard inner product in $\mathbb{R}^2$ by the factor $\cosh^2 v$.

9.2. Find the coefficients of the first fundamental form of

(a) $S^2(1)$ with respect to the local parametrization $\boldsymbol{x}$ defined in Exercise 6.2;

(b) the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x\sin z - y\cos z = 0\}$$

parametrized by

$$\boldsymbol{x}(u,v) = (\sinh v\cos u, \sinh v \sin u, u)$$

Solution:

(a) We have

$$\boldsymbol{x}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

Therefore,

$$\begin{aligned}\partial_u \boldsymbol{x}(u,v) &= \left(\frac{2(1-u^2+v^2)}{(u^2+v^2+1)^2}, \frac{-4uv}{(u^2+v^2+1)^2}, \frac{4u}{(u^2+v^2+1)^2}\right),\\ \partial_v \boldsymbol{x}(u,v) &= \left(\frac{-4uv}{(u^2+v^2+1)^2}, \frac{2(1+u^2-v^2)}{(u^2+v^2+1)^2}, \frac{4v}{(u^2+v^2+1)^2}\right).\end{aligned}$$

Hence,

$$\begin{aligned}
E &= \langle \partial_u \boldsymbol{x} \, , \, \partial_u \boldsymbol{x} \rangle \\
&= \frac{4(1-u^2+v^2)^2 + 16u^2(v^2+1)}{(u^2+v^2+1)^4} \\
&= \frac{4}{(u^2+v^2+1)^2}, \\
F &= \langle \partial_u \boldsymbol{x} \, , \, \partial_v \boldsymbol{x} \rangle \\
&= \frac{-8uv(1-u^2+v^2) - 8uv(1+u^2-v^2) + 18uv}{(u^2+v^2+1)^4} \\
&= 0, \\
G &= \langle \partial_v \boldsymbol{x} \, , \, \partial_v \boldsymbol{x} \rangle \\
&= \frac{4(1+u^2-v^2)^2 + 16v^2(u^2+1)}{(u^2+v^2+1)^4} \\
&= \frac{4}{(u^2+v^2+1)^2}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\partial_u \boldsymbol{x}(u,v) &= (-\sinh v \sin u, \sinh v \cos u, 1), \\
\partial_v \boldsymbol{x}(u,v) &= (\cosh v \cos u, \cosh v \sin u, 0).
\end{aligned}$$

This implies that

$$\begin{aligned}
E(u,v) &= (-\sinh)^2 v \sin^2 u + \sinh^2 v \cos^2 u + 1^2 = \cosh^2 v, \\
F(u,v) &= 0, \\
G(u,v) &= \cosh^2 v \cos^2 u + \cosh^2 v \sin^2 u = \cosh^2 v.
\end{aligned}$$

9.3. Let $U = \mathbb{R} \times (0, \infty)$, and let $\boldsymbol{x} : U \to \mathbb{R}^n$ be a parametrization of a surface $\mathbb{H}$ in $\mathbb{R}^2$ with corresponding coefficients of the first fundamental form $E(u,v) = G(u,v) = 1/v^2$ and $F(u,v) = 0$ for all $(u,v) \in U$. Then $\mathbb{H}$ is called the *hyperbolic plane.* For $r > 0$ denote by $\boldsymbol{\alpha} : (0, \pi) \to \mathbb{H}$ the curve given by

$$\boldsymbol{\alpha}(t) = \boldsymbol{x}(r \cos t, r \sin t).$$

Show that the length of $\boldsymbol{\alpha}$ in $\mathbb{H}$ from $\boldsymbol{\alpha}(\pi/6)$ to $\boldsymbol{\alpha}(5\pi/6)$ is equal to

$$\int_{\pi/6}^{5\pi/6} \frac{1}{\sin t} \, \mathrm{d}t.$$

(In fact, $\boldsymbol{\alpha}$ is the curve of shortest length between its endpoints.) Now take $r = \sqrt{2}$ and find the angle of intersection of $\boldsymbol{\alpha}$ with the curve $\boldsymbol{\beta}(s) = \boldsymbol{x}(1, s)$ at their point of intersection.

Solution:

Let $\boldsymbol{\alpha}(t) = \boldsymbol{x}(c\cos t, c\sin t)$, $\pi/6 \leq t \leq 5\pi/6$ be a curve in the hyperbolic plane. Let $u(t) = c\cos t$ and $v(t) = c\sin t$. The lenght of $\boldsymbol{\alpha}$ is

$$\begin{aligned} l(\boldsymbol{\alpha}) &= \int_{\pi/6}^{5\pi/6} \|\boldsymbol{\alpha}'(t)\| \,\mathrm{d}t \\ &= \int_{\pi/6}^{5\pi/6} (Eu'^2 + 2Fu'v' + Gv'^2)^{1/2} \,\mathrm{d}t \\ &= \int_{\pi/6}^{5\pi/6} \left(\frac{1}{c^2 \sin^2 t} \left(c^2 \sin^2 t + c^2 \cos^2 t\right) \right)^{1/2} \,\mathrm{d}t \\ &= \int_{\pi/6}^{5\pi/6} \frac{1}{\sin t} \,\mathrm{d}t. \end{aligned}$$

If $c = \sqrt{2}$ then $\boldsymbol{\alpha}(t)$ intersects $\boldsymbol{\beta}(s) = \boldsymbol{x}(1, s)$ at points where $u(t) = 1$ and $v(t) = s$. Solving theses equations gives $\cos t = \sin t = \sqrt{2}/2$, so $t = t_0 = \pi/4$ and $s = s_0 = 1$.

At the point of intersection $\partial_u \boldsymbol{x}(1, 1)$, we have $E = G = 1$ and $F = 0$.

At $t_0 = \pi/4$,

$$\begin{aligned} \boldsymbol{\alpha}'(t_0) &= \partial_u \boldsymbol{x}(1,1) u'(t_0) + \partial_v \boldsymbol{x}(1,1) v'(t_0) \\ &= -\sqrt{2} \partial_u \boldsymbol{x}(1,1) \sin t_0 + \sqrt{2} \partial_v \boldsymbol{x}(1,1) \cos t_0 \\ &= -\partial_u \boldsymbol{x}(1,1) + \partial_v \boldsymbol{x}(1,1). \end{aligned}$$

Similarly, at $s_0 = 1$, $\boldsymbol{\beta}'(s_0) = \partial_v \boldsymbol{x}(1, 1)$. Therefore, the angle of intersection of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ at their point of intersection is

$$\cos\vartheta = \frac{\langle \boldsymbol{\alpha}'(t_0), \boldsymbol{\beta}'(s_0) \rangle}{\|\boldsymbol{\alpha}'(t_0)\| \|\boldsymbol{\beta}'(s_0)\|} = \frac{\langle -\partial_u \boldsymbol{x}(1,1) + \partial_v \boldsymbol{x}(1,1), \partial_v \boldsymbol{x}(1,1) \rangle}{\sqrt{1+1}\sqrt{1}} = \frac{1}{\sqrt{2}}.$$

Thus, $\vartheta = \pi/4$.

9.4. Let S be a surface parametrized by

$$\boldsymbol{x}(u, v) = (u\cos v, u\sin v, \log\cos v + u), \qquad (u, v) \in U := \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

For $c \in (-\pi/2, \pi/2)$, let $\boldsymbol{\alpha}_c$ be the curve given by $\boldsymbol{\alpha}_c u = \boldsymbol{x}(u, c)$. Show that the length of $\boldsymbol{\alpha}_c$ from $u = u_0$ to $u = u_1$ does not depend on c.

Solution: The length of $\boldsymbol{\alpha}_c$ is given by

$$l(\boldsymbol{\alpha}_c) = \int_{u_0}^{u_1} \|\boldsymbol{\alpha}_c'(u)\| \,\mathrm{d}u = \int_{u_0}^{u_1} \|\partial_u \boldsymbol{x}(u, c)\| \,\mathrm{d}u = \int_{u_0}^{u_1} \sqrt{E(u, c)} \,\mathrm{d}u$$

We have

$$\partial_u \boldsymbol{x}(u, v) = (\cos v, \sin v, 1),$$

so

$$E = \langle \partial_u \boldsymbol{x}, \partial_u \boldsymbol{x} \rangle = \cos^2 v + \sin^2 v + 1 = 2$$

Thus,

$$l(\boldsymbol{\alpha}_c) = \int_{u_0}^{u_1} \sqrt{2} \,\mathrm{d}u = \sqrt{2}(u_1 - u_0)$$

10.1. Let $\boldsymbol{x} : U \to S$ be a local parametrization of a regular surface S, and denote by E, F, G the coefficients of the first fundamental form in this parametrization. Show that the tangent vector $a\,\partial_u \boldsymbol{x} + b\,\partial_v \boldsymbol{x}$ bisects the angle between the coordinate curves if and only if

$$\sqrt{G}(aE + bF) = \sqrt{E}(aF + bG).$$

Further, if

$$\boldsymbol{x}(u, v) = (u, v, u^2 - v^2),$$

find a vector tangential to S which bisects the angle between the coordinate curves at the point $(1, 1, 0) \in S$.

Solution:

The cosine of the angle of the vector $\boldsymbol{w} = a\,\partial_u \boldsymbol{x} + b\,\partial_v \boldsymbol{x}$ with coordinate curve $v =$ const is equal to

$$\frac{\langle a\,\partial_u \boldsymbol{x} + b\,\partial_v \boldsymbol{x}, \partial_u \boldsymbol{x}\rangle}{\|w\|\|\partial_u \boldsymbol{x}\|} = \frac{aE + bF}{\|w\|\sqrt{E}}$$

Similarly, the cosine of the angle of $\boldsymbol{w}$ with coordinate curve $u =$ const is equal to

$$\frac{\langle a\,\partial_u \boldsymbol{x} + b\,\partial_v \boldsymbol{x}, \partial_v \boldsymbol{x}\rangle}{\|w\|\|\partial_v \boldsymbol{x}\|} = \frac{aF + bG}{\|w\|\sqrt{G}}$$

The equality of the cosines

$$\frac{aE + bF}{\|w\|\sqrt{E}} = \frac{aF + bG}{\|w\|\sqrt{G}}$$

is equivalent to

$$\sqrt{G}(aE + bF) = \sqrt{E}(aF + bG)$$

as required.

For

$$\boldsymbol{x}(u, v) = (u, v, u^2 - v^2),$$

we have

$$\begin{aligned} \partial_u \boldsymbol{x}(u, v) &= (1, 0, 2u), \\ \partial_v \boldsymbol{x}(u, v) &= (0, 1, -2v), \end{aligned}$$

which implies that

$$E(u, v) = 1 + 4u^2, \qquad F(u, v) = -4uv, \qquad G(u, v) = 1 + 4v^2.$$

The point $(1, 1, 0)$ has coordinates $(u, v) = (1, 1)$, so we have $E = G = 5$, $F = -4$. Thus, we obtain the following equation on (a, b):

$$\sqrt{5}(5a - 4b) = \sqrt{5}(-4a + 5b),$$

which is equivalent to $a = b$. Thus, the vector $\partial_u \boldsymbol{x} + \partial_v \boldsymbol{x}$ bisects the angle.

10.2. Find two families of curves on the helicoid parametrized by

$$\boldsymbol{x}(u, v) = (v \cos u, v \sin u, u)$$

which, at each point, bisect the angles between the coordinate curves.

(Show that they are given by $u \pm \sinh^{-1} v = c$, where c is a constant on each curve in the family.)

Solution: We have

$$\begin{aligned}\partial_u \boldsymbol{x}(u,v) &= (-v\sin u, v\cos u, 1),\\ \partial_v \boldsymbol{x}(u,v) &= (\cos u, \sin u, 0),\end{aligned}$$

which implies that

$$E(u,v) = 1+v^2, \qquad F(u,v)=0, \qquad G(u,v)=1,$$

so the equation from Exercise 10.1 becomes

$$a\sqrt{v^2+1} = b.$$

The curve $u - \sinh^{-1} v = c$ can be parametrized by $\boldsymbol{\alpha}(u) = (u, \sinh(u-c))$, so

$$\boldsymbol{\alpha}'(u,v) = \partial_u \boldsymbol{x} + \cosh(u-c)\partial_v \boldsymbol{x} = \partial_u \boldsymbol{x} + \cosh(u-c)\partial_v \boldsymbol{x} = \partial_u \boldsymbol{x} + \sqrt{v^2+1}\,\partial_v \boldsymbol{x}$$

as required.

The curve $u + \sinh^{-1} v = c$ can be parametrized by $\boldsymbol{\beta}(u) = (u, -\sinh(u-c))$, so

$$\boldsymbol{\beta}'(u,v) = \partial_u \boldsymbol{x} - \cosh(u-c)\partial_v \boldsymbol{x} = \partial_u \boldsymbol{x} - \cosh(u-c)\partial_v \boldsymbol{x} = \partial_u \boldsymbol{x} - \sqrt{v^2+1}\,\partial_v \boldsymbol{x}.$$

Then

$$\langle \boldsymbol{\alpha}', \boldsymbol{\beta}' \rangle = E - (v^2+1)G = 0,$$

which implies that $\boldsymbol{\beta}'$ bisects the angle between $\partial_u \boldsymbol{x}$ and $-\partial_v \boldsymbol{x}$.

10.3. The coordinate curves of a parametrization $\boldsymbol{x}(u,v)$ constitute a *Chebyshev net* if the lengths of the opposite sides of any quadrilateral formed by them are equal.

(a) Show that a necessary and sufficient condition for this is

$$\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0.$$

(b) Show that if coordinate curves constitute a Chebyshev net, then it is possible to reparametrize the coordinate neighborhood in such a way that the new coefficients of the first fundamental form are

$$E = 1, \qquad F = \cos\vartheta, \qquad G = 1,$$

where ϑ is the angle between coordinate curves.

Solution:

(a) Assume that coordinate curves constitute a Chebyshev net. Consider a quadrilateral with vertices $(u_0,v_0),(u_1,v_0),(u_0,v_1),(u_1,v_1)$ formed by coordinate curves. The length of the side with vertices $(u_0,v_1),(u_1,v_1)$ is equal to

$$\int_{u_0}^{u_1} \|\partial_u \boldsymbol{x}(u,v_1)\|\,\mathrm{d}u = \int_{u_0}^{u_1} \sqrt{E(u,v_1)}\,\mathrm{d}u$$

Thus, the integral $\int_{u_0}^{u_1} \sqrt{E(u,v_1)}\,\mathrm{d}u$ does not depend on v_1, i.e. it is a function of u_1 only. Differentiating it by u_1, we see that $\sqrt{E(u_1,v_1)}$ is also a function of u_1 only, so $E(u,v)$ does not depend on v. The considerations for G are similar, and the converse statement is straightforward.

(b) Take

$$\tilde{u}(u) = \int \sqrt{E(u)}\,\mathrm{d}u$$

Then $\tilde{u}$ is parametrized by arc length, so $\tilde{E}(\tilde{u}) \equiv 1$. Similarly, we can make $\tilde{G}(\tilde{v}) \equiv 1$. Now F is equal to the cosine of the angle by definition.

10.4. Show that a surface of revolution can always be parametrized so that

$$E = E(v), \qquad F = 0, \qquad G = 1$$

Solution: Parametrize the surface by

$$\boldsymbol{x} = (f(v)\cos u, f(v)\sin u, g(v)),$$

where $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$ is the generating curve. Then

$$\begin{aligned} \partial_u \boldsymbol{x} &= (-f(v)\sin u, f(v)v\cos u, 0), \\ \partial_v \boldsymbol{x} &= (f'(v)\cos u, f'(v)\sin u, g'(v)), \end{aligned}$$

which implies that

$$E(u,v) = f^2(v), \qquad F(u,v) = 0, \qquad G(u,v) = f'^2(v) + g'^2(v) = \|\boldsymbol{\alpha}'\|^2$$

Parametrizing $\boldsymbol{\alpha}(v)$ by arc length we obtain a required parametrization of the surface.

10.5. Let S be the surface $\{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}$ and let $\mathcal{F}$ be the family of curves on S obtained as the intersection of S with the planes $z = \text{const}$. Find the family of curves on S which meet $\mathcal{F}$ orthogonally and show that they are the intersections of S with the family of hyperbolic cylinders $xy = \text{const}$.

Solution:

A (part of a) curve $x^2 - y^2 = c_1$ on S can be parametrized by $\boldsymbol{\alpha}(y) = (\sqrt{y^2+c}, y)$, so

$$\boldsymbol{\alpha}'(y) = \frac{y}{\sqrt{y^2+c_1}}\partial_x + \partial_y = \frac{y}{x}\partial_x + \partial_y$$

A curve $xy = c_2$ on S can be paranetrized by $\boldsymbol{\beta}(x) = (x, \frac{c}{x})$, so

$$\boldsymbol{\beta}'(x) = \partial_x - \frac{c_2}{x^2}\partial_y = \partial_x - \frac{y}{x}\partial_y$$

Now we recall that the coefficients of the first fundamental form found in Exercise 10.1 are

$$E(u,v) = 1 + 4x^2, \qquad F(u,v) = -4xy, \qquad G(u,v) = 1 + 4y^2,$$

so we compute the inner product of $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ to get

$$\begin{aligned} \left\langle \boldsymbol{\alpha}', \boldsymbol{\beta}' \right\rangle = \left\langle \frac{y}{x}\partial_x + \partial_y, \partial_x - \frac{y}{x}\partial_y \right\rangle &= \frac{y}{x}E + F - \frac{y^2}{x^2}F - \frac{y}{x}G = \\ &= \frac{y}{x}(1+4x^2) + 4xy\left(\frac{y^2}{x^2} - 1\right) - \frac{y}{x}(1+4y^2) = \frac{y}{x} + 4xy + \frac{4y}{x} - 4xy - \frac{y}{x} - \frac{4y}{x} = 0 \end{aligned}$$

Note that we could avoid computations on S: one could consider $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as curves in $\mathbb{R}^3$, and keeping in mind that z-coordinate of $\boldsymbol{\alpha}'$ is equal to zero, the dot product of $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ is equal to $\big\langle \big(\frac{y}{x}, 1, 0\big) \cdot \big(1, -\frac{y}{x}, z'(x)\big) \big\rangle = 0$.

10.6. Using the notation of Exercise 10.2, show that the family of curves orthogonal to the family

$$v \cos u = \text{const}$$

is the family defined by $(1+v^2)\sin^2 u = \text{const}$.

Solution:

The coefficients of the first fundamental form found in Exercise 10.2 are

$$E(u,v) = 1+v^2, \qquad F(u,v) = 0, \qquad G(u,v) = 1.$$

A curve $v\cos u = c_1$ on S can be parametrized by $\boldsymbol{\alpha}(u) = (u, c_1/\cos u)$, so

$$\boldsymbol{\alpha}'(u) = (1, -c_1 \sin u/\cos^2 u) = (1, -v\tan u).$$

A curve $(1+v^2)\sin^2 u = c_2$ on S can be paranetrized by $\boldsymbol{\beta}(u) = \left(u, -\sqrt{\frac{c_2}{\sin^2 u} - 1}\right)$, so

$$\boldsymbol{\beta}'(u) = \left(1, \frac{1}{\tan u}\left(v + \frac{1}{v}\right)\right).$$

Computing the inner product of $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ we obtain

$$\langle \boldsymbol{\alpha}', \boldsymbol{\beta}' \rangle = E - v\tan u \frac{1}{\tan u}\left(v + \frac{1}{v}\right) = 1 + v^2 - (v^2+1) = 0.$$