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Differential Geometry 3.12.2019

Homework 9-10

First fundamental form

9.1. Find the coefficients of the first fundamental forms of:

(a) the *catenoid* parametrized by

$$\boldsymbol{x}(u,v) = (\cosh v \cos u, \cosh v \sin u, v), \qquad (u,v) \in U := (0,2\pi) \times \mathbb{R};$$

(b) the *helicoid* parametrized by

$$\widetilde{\boldsymbol{x}}(u,v) = (-\sinh v \sin u, \sinh v \cos u, -u), \qquad (u,v) \in U;$$

(c) the surface S_ϑ (for some $\vartheta \in \mathbb{R}$) parametrized by

$$\boldsymbol{y}_\vartheta(u,v) = (\cos\vartheta)\boldsymbol{x}(u,v) + (\sin\vartheta)\widetilde{\boldsymbol{x}}(u,v), \qquad (u,v) \in U.$$

9.2. Find the coefficients of the first fundamental form of $S^2(1)$ with respect to the local parametrization $\boldsymbol{x}$ defined in Exercise 6.2.

9.3. Let $U = \mathbb{R} \times (0,\infty)$, and let $\boldsymbol{x} : U \to \mathbb{R}^n$ be a parametrization of a surface $\mathbb{H}$ in $\mathbb{R}^2$ with corresponding coefficients of the first fundamental form $E(u,v) = G(u,v) = 1/v^2$ and $F(u,v) = 0$ for all $(u,v) \in U$. Then $\mathbb{H}$ is called the *hyperbolic plane*. For $r > 0$ denote by $\boldsymbol{\alpha} : (0,\pi) \to \mathbb{H}$ the curve given by

$$\boldsymbol{\alpha}(t) = \boldsymbol{x}(r\cos t, r\sin t).$$

Show that the length of $\boldsymbol{\alpha}$ in $\mathbb{H}$ from $\boldsymbol{\alpha}(\pi/6)$ to $\boldsymbol{\alpha}(5\pi/6)$ is equal to

$$\int_{\pi/6}^{5\pi/6} \frac{1}{\sin t}\, \mathrm{d}t.$$

(In fact, $\boldsymbol{\alpha}$ is the curve of shortest length between its endpoints.) Now take $r = \sqrt{2}$ and find the angle of intersection of $\boldsymbol{\alpha}$ with the curve $\boldsymbol{\beta}(s) = \boldsymbol{x}(1,s)$ at their point of intersection.

9.4. Let S be a surface parametrized by

$$\boldsymbol{x}(u,v) = (u\cos v, u\sin v, \log\cos v + u), \qquad (u,v) \in U := \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

For $c \in (-\pi/2, \pi/2)$, let $\boldsymbol{\alpha}_c$ be the curve given by $\boldsymbol{\alpha}_c(u) = \boldsymbol{x}(u,c)$. Show that the length of $\boldsymbol{\alpha}_c$ from $u = u_0$ to $u = u_1$ does not depend on c.

Coordinate curves, angles and area

10.1. Let $\boldsymbol{x} : U \to S$ be a local parametrization of a regular surface S, and denote by E, F, G the coefficients of the first fundamental form in this parametrization. Show that the tangent vector $a\,\partial_u \boldsymbol{x} + b\,\partial_v \boldsymbol{x}$ bisects the angle between the coordinate curves if and only if

$$\sqrt{G}(aE + bF) = \sqrt{E}(aF + bG).$$

Further, if

$$\boldsymbol{x}(u, v) = (u, v, u^2 - v^2),$$

find a vector tangential to S which bisects the angle between the coordinate curves at the point $(1, 1, 0) \in S$.

10.2. Find two families of curves on the helicoid parametrized by

$$\boldsymbol{x}(u, v) = (v \cos u, v \sin u, u)$$

which, at each point, bisect the angles between the coordinate curves.

(Show that they are given by $u \pm \sinh^{-1} v = c$, where c is a constant on each curve in the family.)

10.3. The coordinate curves of a parametrization $\boldsymbol{x}(u, v)$ constitute a *Chebyshev net* if the lengths of the opposite sides of any quadrilateral formed by them are equal.

(a) Show that a necessary and sufficient condition for this is

$$\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$$

(b) Show that if coordinate curves constitute a Chebyshev net, then it is possible to reparametrize the coordinate neighborhood in such a way that the new coefficients of the first fundamental form are

$$E = 1, \qquad F = \cos\vartheta, \qquad G = 1,$$

where ϑ is the angle between coordinate curves.

10.4. Show that a surface of revolution can always be parametrized so that

$$E = E(v), \qquad F = 0, \qquad G = 1$$

10.5. Let S be the surface $\{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}$ and let $\mathcal{F}$ be the family of curves on S obtained as the intersection of S with the planes $z = \text{const}$. Find the family of curves on S which meet $\mathcal{F}$ orthogonally and show that they are the intersections of S with the family of hyperbolic cylinders $xy = \text{const}$.

10.6. Using the notation of Exercise 10.2, show that the family of curves orthogonal to the family

$$v \cos u = \text{const}$$

is the family defined by $(1 + v^2)\sin^2 u = \text{const}$.