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Solutions for questions from Problem Class 8 (schduled to 20.03.2020)

1. The aim of this question is to verify the Gauss-Bonnet theorem for a region R on the surface S given by the local parametrisation $x(u, v) = (v \cos u, v \sin u, v^2)$, where the region R is defined by $0 \leq u \leq 2\pi$, $0 \leq v < 1$.

   (a) State the global Gauss-Bonnet Theorem.

   (b) Compute the coefficients of the first and second fundamental forms on S.

   (c) Compute Gauss curvature K, calculate $\int_R K dA$.

   (d) Show that the curve $\gamma(u) = x(u, 1)$ is unit speed. Find the geodesic curvature κ_g and compute $\int_{\partial R} \kappa_g ds$.

   (e) Compute the Euler characteristic $\chi(R)$ of the region R. Verify the Gauss-Bonnet theorem for the region R.

Solution:

(a) Let R be a region in an oriented surface S. Then

$$\int_R K \,\mathrm{d}A + \int_{\partial R} \kappa_g \,\mathrm{d}s + \sum_{j=1}^{r} \theta_j = 2\pi\chi(R).$$

(b) As $\boldsymbol{x}_u = (-v \sin u, v \cos u, 0)$ and $\boldsymbol{x}_v = (\cos u, \sin u, 2v)$, we have :

$$E = v^2, \quad F = 0 \quad G = 1 + 4v^2.$$

(You can either compute it explicitely, or to use that for a surface of revolution with parametrisation $\boldsymbol{x}(u, v) = (f(v) \cos u, f(v) \sin u, g(v))$ one gets $E = f^2$, $F = 0$, $G = f'^2 + g'^2$.)

Next, we have $||\boldsymbol{x}_u \times \boldsymbol{x}_v|| = \sqrt{EG - F^2} = v\sqrt{1 + 4v^2}$ and

$$\boldsymbol{N} = \frac{1}{\sqrt{1 + 4v^2}}(2v^2 \cos u, 2v^2 \sin u, -v) = \frac{(2v \cos u, 2v \sin u, -1)}{\sqrt{1 + 4v^2}}.$$

Also,

$$\boldsymbol{x}_{uu} = (-v \cos u, -v \sin u, 0), \quad \boldsymbol{x}_{uv} = (-\sin u, \cos u, 0), \quad -\boldsymbol{x}_{vv} = (0, 0, 2),$$

and we have

$$L = -\frac{2v^2}{\sqrt{1 + 4v^2}}, \quad M = 0, \quad N = -\frac{2}{\sqrt{1 + 4v^2}}.$$

(c) The region R is a lower part of this paraboloid of revolution (looks as a tea cup).
We have

$$K = \frac{LN - M^2}{EG - F^2} = \frac{4v^/(1 + 4v^2)}{v^2(1 + 4v^2)} = \frac{4}{(1 + 4v^2)^2}.$$

and

$$\mathrm{d}A = \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v = v\sqrt{1 + 4v^2}\,\mathrm{d}u\,\mathrm{d}v.$$

So, we get

$$\int_R K \,\mathrm{d}A = \int_0^{2\pi} \mathrm{d}u \int_0^1 \frac{4}{(1 + 4v^2)^2} \cdot v\sqrt{1 + 4v^2}\,\mathrm{d}v = 2\pi \int_0^1 \frac{4v}{(1 + 4v^2)^{3/2}}\,\mathrm{d}v =$$
$$-2\pi \int_0^1 ((1 + 4v^2)^{-1/2})'\,\mathrm{d}v = -2\pi \frac{1}{\sqrt{1 + 4v^2}}\Big|_0^1 = -2\pi(\frac{1}{\sqrt{5}} - 1) = 2\pi(1 - \frac{1}{\sqrt{5}}).$$

(d) As $\boldsymbol{\gamma}(u) = \boldsymbol{x}(u,1) = (\cos u, \sin u, 1)$, we have $\boldsymbol{\gamma}'(u) = \boldsymbol{x}_u(u,1) = (-\sin u, \cos u, 0)$, so $\boldsymbol{\gamma}$ is a unit speed curve. Therefore, we can compute the geodesic curvature as follows:

$$k_g = (\boldsymbol{\gamma}' \times \gamma'') \cdot \boldsymbol{N}.$$

Since $\boldsymbol{\gamma}'(u) = (-\sin u, \cos u, 0)$ and $\boldsymbol{\gamma}''(u) = (-\cos u, -\sin u, 0)$, we get $\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'' = (0,0,1)$. (this makes sense as $\boldsymbol{\gamma}$ is a plane curve!).
Plugging $v = 1$ into the expression for $\boldsymbol{N}$, we get

$$\boldsymbol{N} = \frac{1}{\sqrt{5}}(2\cos u, 2\sin u, -1).$$

Hence, $k_g = (\boldsymbol{\gamma}' \times \gamma'') \cdot \boldsymbol{N} = -\frac{1}{\sqrt{5}}$.
To compute $\int_{\partial R} \kappa_g \, \mathrm{d}s$ we need to choose the correct orientation of the curve $\boldsymbol{\gamma}$, which means, when we walk on the surface slong the boundary the the region stays on the left. Notice, that the normal $\boldsymbol{N}$ has negative third coordinate, so it is looking downwards (i.e. the surfaces is *outside* of the paraboloid. Walking the boundary of R from outside of the paraboloid and so, that the surface is on the left is following the circle $(\cos u, \sin u, 1)$ from $u = 2\pi$ to $u = 0$. Hence,

$$\int_{\partial R} \kappa_g \, \mathrm{d}s = \int_{u=2\pi}^{0} (-\frac{1}{\sqrt{5}}) \, \mathrm{d}u = \frac{2\pi}{\sqrt{5}}.$$

(e) The region R is homeomorphic to a disc, so $\chi(R) = 1$, also, there are no vertices on the boundary (the boundary is smooth), so that $\sum_{j=1}^{r} \theta_j = 0$. Hence, veryfying Gauss-Bonnet theorem is equivalent to checking that

$$\int_R K \, \mathrm{d}A + \int_{\partial R} \kappa_g \, \mathrm{d}s = 2\pi,$$

which is true:

$$(2\pi - 2\pi\frac{1}{\sqrt{5}}) + \frac{2\pi}{\sqrt{5}} = 2\pi.$$

2. Oriented closed surfaces of constant curvature:

It is known that an oriented closed surface is a sphere with a non-negative integer number of handles (a sphere with g handles is called a surface of genus g and will be denoted by S_g).

(a) (was not planned for discussion in the Problems Class) Show by induction on g that $\chi(S_g) = 2-2g$.

(b) Suppose that S_g is a surface of constant curvature K. Show that
- if $g = 0$ then $K > 0$;
- if $g = 1$ then $K = 0$;
- if $g > 1$ then $K < 0$.

Solution for (b): If S is a closed surface, then it has no boundary and no boundary vertices, so that Gauss-Bonnet theorem is reduced to

$$\int_R K \, \mathrm{d}A = 2\pi(2 - 2g).$$

In assumption of constant curvature, this further reduces to

$$K \cdot area(S) = 2\pi(2 - 2g).$$

As area is positive, this imlies that when $g = 0$ we get $K > 0$, when $g = 1$ we get $K = 0$, and otherwise $K < 0$.