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Revision

1. Regular curves:
- parametrise a curve, show it is regular ($\boldsymbol{\alpha}' \neq 0$);
- compute length, arc-length parametrization ($l(u) = \int_{u_0}^{u} ||\boldsymbol{\alpha}'(v)||\,\mathrm{d}v$) ;
- find curvature and torsion ($\kappa(u) = \frac{||\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''||}{||\boldsymbol{\alpha}'||^3}$, $\tau(u) = \frac{-(\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'')\cdot\boldsymbol{\alpha}'''}{||\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''||^2}$), for a plane curve $\kappa(u) = \frac{x'y''-x''y'}{((x')^2+(y')^2)^{3/2}}$;
- find vertices ($\kappa'(u_0) = 0$) and inflection points ($\kappa(u_0) = 0$), 4-vertex theorem (for plane curves)
- find centre of curvature, evolute $\boldsymbol{e}(s) = \boldsymbol{\alpha}(s) + \frac{1}{\kappa(s)}\boldsymbol{n}(s)$ and involute (for plane curves);
- Serret-Frenet equations: $\boldsymbol{t}' = \kappa\boldsymbol{n}$

$$\boldsymbol{n}' = -\kappa\boldsymbol{t} - \tau\boldsymbol{b}$$
$$\boldsymbol{b}' = \tau\boldsymbol{n}$$

- Fundamental theorems of local theory of plane and space curves.

2. Surfaces:
- Parametrized regular surfaces ($\boldsymbol{x}(u,v)$ smooth, homeo, $\boldsymbol{x}_u, \boldsymbol{x}_v$ linear indep.);
- graphs and level sets, Implicit Function Theorem,
- FFF ($E = \langle\boldsymbol{x}_u, \boldsymbol{x}_u\rangle, F = \langle\boldsymbol{x}_u, \boldsymbol{x}_v\rangle, G = \langle\boldsymbol{x}_v, \boldsymbol{x}_v\rangle$), area;
- special surfaces (surfaces of revolution, ruled surfaces, canal surfaces).

3. Maps beween surface:
- Gauss map $\boldsymbol{N} \circ \boldsymbol{x}(u,v) = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{||\boldsymbol{x}_u \times \boldsymbol{x}_v||}(u,v)$;
- isometries, conformal maps $\langle d_p f(\boldsymbol{w}_1), d_p f(\boldsymbol{w}_2)\rangle_{f(p)} = \lambda(p)\langle\boldsymbol{w}_1, \boldsymbol{w}_2\rangle_{\boldsymbol{p}}$.

4. Weingarten map ($-d_p\boldsymbol{N}$)
- principle curvatures κ_1, κ_2 - eigenvalues of $-d_p\boldsymbol{N}$;
- Gauss and mean curvatures $K = \kappa_1\kappa_2$, $H = \frac{1}{2}(\kappa_1 + \kappa_2)$;
- Second Fundamental Form $\langle -d_p\boldsymbol{N}(w), w\rangle$, $L = \boldsymbol{x}_{uu}\cdot\boldsymbol{N}$, $M = \boldsymbol{x}_{uv}\cdot\boldsymbol{N}$, $N = \boldsymbol{x}_{vv}\cdot\boldsymbol{N}$;
- umbilic points ($\kappa_1 = \kappa_2$), elliptic, hyperbolic, flat points ($K > 0$, $K < 0$, $K = 0$ resp.);
- minimal surfaces ($H = 0$);
- $-d_p\boldsymbol{N} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$, $K = \frac{LN-M^2}{EG-F^2}$;
- principal parametrization: $M = F = 0$. Then $\kappa_1 = \frac{L}{E}$, $\kappa_2 = \frac{N}{G}$.

5. Gauss Theorema Egregium
- calculation of K of an abstract surface given by $U \subset \mathbb{R}^2$ and FFF.

6. Curves on surfaces
- Coordinate curves $u = \textit{const}$ or $v = \textit{const}$;
- Geodesic and normal curvature $\boldsymbol{\alpha}'' = \kappa_g(\mathcal{N} \times \boldsymbol{\alpha}') + \kappa_n\boldsymbol{N}$;
- Meusnier Theorem $\kappa_n(s) = II_p(\frac{\boldsymbol{w}}{||\boldsymbol{w}||})$; $\kappa_g = \frac{(\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'')\cdot\boldsymbol{N}}{||\boldsymbol{\alpha}'||^3}$;
- lines of curvature (tangents are principal directions);
- asymptotic curves $\kappa_n = 0$;
- geodesics $\boldsymbol{\alpha}''(s) \perp T_{\boldsymbol{\alpha}(s)}S$, local existence and uniqueness, isometries preserve geodesics.

7. Gauss-Bonnet Theorem $\int_R K\,\mathrm{d}A + \int_{\partial R}\kappa_g\,\mathrm{d}s + \sum_{j=1}^{r}\theta_j = 2\pi\chi(R)$.