14 Hilbert's Theorem

Non-examinable section!

This is what I was planning to tell in the two lecture of Week 20. However, it is not examinable, and it is an additional part of the course which is only added recently (as Epiphany term is now scheduled to be 1 week longer than it was before).

You can look through this material if you like, in particular, if you want to see how can one use some of the notions we were looking at.

We will closely follow section 5-11 from Do Carmo book (and I need to advise you that the book is a much better reading than my notes, while at the same time I have tried to simplify the things to bring them closer to the course).

Definition 14.1. A surface S is *immersed* in $\mathbb{R}^3$ if it is smoothly embedded to $\mathbb{R}^3$, possibly with self-intersections, i.e. if for every point $p \in S$ there is an open set $U \in \mathbb{R}^2$ and an injective local diffeomorphism $x : U \to S$ such $p \in x(U)$.

Remark 14.2. If you dislike this definition, you can just think about ordinary embedding instead.

The aim of these notes is to prove the following theorem (which is a bit weaker than the general version of Hilbert's theorem, see Theorem 14.12 below).

Theorem 14.3. *There is no isometric immersion of a complete hyperbolic plane to* $\mathbb{R}^3$.

Remark 14.4. By complete space here we mean "complete as a metric space", i.e. every Cauchy sequence is converging in the space).

Remark 14.5. By hyperbolic plane $\mathbb{H}^2$ we mean the upper half-plane $\{(u, v) \in \mathbb{R}^2 \mid v > 0\}$ with the First Fundamental Form on it given by $E = 1/v^2 = G$, $F = 0$.

Lemma 14.6. *The area of* $\mathbb{H}^2$ *is infinite.*

To prove the lemma, one can either directly integrate or to recall from Example 13.13 that the area of a hyperbolic triangle with vertices at -1, 1, and ∞ equals to π and to observe that $\mathbb{H}^2$ contains infinitely many isometric copies of this triangle (in particular, the triangles with vertices at $2k - 1$, $2k + 1$, ∞ for $k \in \mathbb{Z}$).

For the proof of Theorem 14.3 we will need the notion of *Chebyshev net*:

Definition 14.7. Let x be a parametrisation of S. Coordinate curves of the parametrisation constitute a *Chebyshev net* if the opposite sides of any quadrilateral formed by coordinate curves are of equal length.

Idea of proof of Hilbert's Theorem.

Step 0. Let $S = \mathbb{H}^2$ and assume that $\varphi : S \to \mathbb{R}^3$ is an isometric immersion.

Step 1. We will use φ to find a parametrisation x of S such that coordinate curves of x

(1) are asymptotic curves of S and

(2) form a Chebyshev net.

(See Proposition 14.9.)

Step 2. We will show that an area of any quadrilateral formed by coordinate curves is smaller than 2π. (See Proposition 14.10.)

Step 3. We will cover S be a sequence of nested growing quadrilaterals of bounded area and will conclude that the area of S is finite (in contradiction to Lemma 14.6).

We will start with mapping the hyperbolic plane isometrically to $\mathbb{R}^2$ (this is possible as open helpf-plane is homeomorphis to the plane), the image of that map will be called S.

14.1 Step 1: constructing Chebyshev net

We will use the following statement about a Chebyshev net:

Lemma 14.8 (Excercise 7, Section 2-5 in Do Carmo). *Suppose that x is a local parametrisation of a surface S such that $E_v = G_u = 0$. Then the coordinate curves of this parametrisation form a Chebyshev net.*

Proof. Consider the quadrilateral formed by the coordinate curves $u = 0$, $u = \epsilon$ and $v = 0$,$v = \delta$. We need to show that the length l_1 of the coordinate curve $u = 0$ between $(0, 0)$ and $(0, \delta)$ is the same as the length l_2 of the coordinate curve $u = \epsilon$ between $(\epsilon, 0)$ and (ϵ, δ). In other words, that

$$l_1 = \int_0^\delta \sqrt{\langle x'(0,t), x'(0,t)\rangle}\, \mathrm{d}t = \int_0^\delta \sqrt{\langle x'(\epsilon,t), x'(\epsilon,t)\rangle}\, \mathrm{d}t = l_2.$$

At the same time $\langle x'(\epsilon, t), x'(\epsilon, t)\rangle = \sqrt{G(\epsilon, t)}$. As $G_u = 0$, we conclude that $G(0, t) = G(\epsilon, t)$, and hence, $l_1 = l_2$.

Similarly we can use E_v to conclude the equal length for the other pair of the sides. Notice also that we can do the same for coordinate quadrilaterals not containing the point $(0, 0)$.

□

To perform Step 1, we will prove the following proposition:

Proposition 14.9 (Lemma 2 in Do Carmo). *Suppose there is an isometric immersion $\varphi : S = \mathbb{H}^2 \to \mathbb{R}^3$. Then for every point $p \in S$ there exists a parametrisation $x : U \in \mathbb{R} \to S$, $p \in X(U)$ such that the coordinate curves of x are asymptotic curves of $x(U) = V$ and form a Chebyshev net.*

Proof. Recall from Remark 11.11(d) that is $K < 0$ then for every point $p \in S$ there exists an open set $V_p \in S$, $p \in V$ and a local parametrisation x in V such that the coordinate curves are asymptotic curves and the second fundamental form has $L = N = 0$, i.e. $II_p = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}$.

By Lemma 14.9, it is now sufficient to prove that for the parametrisation x one has $E_v = G_v = 0$: indeed, in this parametrisation the coordinate curves are asymptotic curves and Lemma 14.9 will provide the Chebyshev net property.

The proof of $E_v = G_v = 0$ will be a bit technical - but this is exactly the point where we use existence of isometric immersion to $\mathbb{R}^3$. Namely, considering this isometric immersion, we can look at the normal to the surface, and in particular to compute that

$$\boldsymbol{N}_u \times \boldsymbol{N}_v = d\boldsymbol{N}_p(\boldsymbol{x}_u) \times d\boldsymbol{N}_p(\boldsymbol{x}_v) \overset{(A)}{=} det(d\boldsymbol{N}_p)(\boldsymbol{x}_u \times \boldsymbol{x}_v) \overset{def K}{=} K\boldsymbol{x}_u \times \boldsymbol{x}_v \overset{(B)}{=} K\boldsymbol{N}\sqrt{EG - F^2} = KD\boldsymbol{N}, \quad (1)$$

where

- in the equality labelled (A), we use that the absolute values are equal as areas of corresponding parallelograms are the same and the vectors $\boldsymbol{N}_u \times \boldsymbol{N}_v$ and $\boldsymbol{x}_u \times \boldsymbol{x}_v$ are parallel since $\boldsymbol{x}_u \times \boldsymbol{x}_v$ is parallel to $\boldsymbol{N}$ (obviously), and $\boldsymbol{N}_u \times \boldsymbol{N}_v$ is also parallel to $\boldsymbol{N}$ since by differentiating $\langle \boldsymbol{N}, \boldsymbol{N}\rangle = 1$ we get $\langle \boldsymbol{N}_u, \boldsymbol{N}\rangle = 0$ and $\langle \boldsymbol{N}_v, \boldsymbol{N}\rangle = 0$.
- the equality labelled (B) is just that $\boldsymbol{N} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{||\boldsymbol{x}_u \times \boldsymbol{x}_v||} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\sqrt{EG-F^2}}$.
- the last equality is the standard notation $D = \sqrt{EG - F^2}$.

Next, we have

$$(\boldsymbol{N} \times \boldsymbol{N}_u)_v = \boldsymbol{N}_v \times \boldsymbol{N}_u + \boldsymbol{N} \times \boldsymbol{N}_{uv} \quad \text{and} \quad (\boldsymbol{N} \times \boldsymbol{N}_v)_u = \boldsymbol{N}_u \times \boldsymbol{N}_v + \boldsymbol{N} \times \boldsymbol{N}_{uv},$$

which implies

$$(\boldsymbol{N} \times \boldsymbol{N}_u)_v - (\boldsymbol{N} \times \boldsymbol{N}_v)_u = -2\boldsymbol{N}_u \times \boldsymbol{N}_v \overset{(1)}{=} -2KD\boldsymbol{N}. \quad (2)$$

Furthermore,

$$\boldsymbol{N} \times \boldsymbol{N}_u = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{||\boldsymbol{x}_u \times \boldsymbol{x}_v||} \times \boldsymbol{N}_u = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{D} \times \boldsymbol{N}_u = \frac{1}{D}(\langle \boldsymbol{x}_u, \boldsymbol{N}_u\rangle \boldsymbol{x}_v - \langle \boldsymbol{x}_v, \boldsymbol{N}_u\rangle \boldsymbol{x}_u) \overset{(C)}{=} \frac{1}{D}(M\boldsymbol{x}_u - L\boldsymbol{x}_v),$$

where the second before last equation is a general property of vector products and the last equation (the one labelled (C)) holds since by differentiating $\langle \boldsymbol{N}, \boldsymbol{X}_u\rangle = 0$ we get that $L = \langle \boldsymbol{N}, \boldsymbol{x}_{uu}\rangle = -\langle \boldsymbol{x}_u, \boldsymbol{N}_u\rangle$ and similarly, $M = \langle \boldsymbol{x}_u, \boldsymbol{N}_u\rangle$.
Similarly,

$$\boldsymbol{N} \times \boldsymbol{N}_v = \frac{1}{D}(N\boldsymbol{x}_u - M\boldsymbol{x}_v).$$

Now, recall from the first lines of the proof that $L = N = 0$, and also recall that we assume $K = -1$, and hence, $K = -1 = \frac{LN-M^2}{D^2} = -\frac{M^2}{D^2}$, which implies $M = \pm D$.

Therefore,

$$\boldsymbol{N} \times \boldsymbol{N}_u = \pm \boldsymbol{x}_u \quad \text{and} \quad \boldsymbol{N} \times \boldsymbol{N}_v = \pm \boldsymbol{x}_v,$$

which by equation (2) implies that

$$2KD\boldsymbol{N} \overset{K=-1}{=} -2D\boldsymbol{N} \overset{(2)}{=} \pm \boldsymbol{x}_{uv} \pm \boldsymbol{x}_{vu} = \pm 2\boldsymbol{x}_{uv}. \tag{3}$$

Notice that the sign $\pm$ is chosen simultaneously everywhere (coming from $M = \pm D$), so in the last equation we get $\pm 2\boldsymbol{X}_{uv}$ and not 0 indeed.

The equation (3) implies that $\boldsymbol{x}_{uv}$ is parallel to $\boldsymbol{N}$. So, using .... we get

$$E_v = 2\langle \boldsymbol{x}_{uv}, \boldsymbol{x}_u \rangle = 0 \quad \text{and} \quad G_u = 2\langle \boldsymbol{x}_{uv}, \boldsymbol{x}_v \rangle = 0,$$

as required. □

14.2 Step 2: Finite area of coordinate quadrilaterals.

Proposition 14.10. *Suppose that the coordinate curves form a Chebyshev net. Then*

(a) there exists a parametrisation of the coordinate neighbourhood such that $E = 1 = G$ and $F = \cos\theta$ where θ is the angle between the coordinate curves.

(b) for the parametrisation described in (a) one has $K = -\frac{\theta_{uv}}{\sin\theta}$.

Proof. (a) Since the coordinate curves form a Chebyshev net, we immediately get that $E_v = G_u = 0$ (this is similar to the proof of Lemma 14.9). Hence, $E(u, v) = E(u)$ is a function of u alone. Similarly, $G = G(v)$. Set

$$\widetilde{u} = \int_0^u \sqrt{E(t)}\,\mathrm{d}t \quad \text{and} \quad \widetilde{v} = \int_0^v \sqrt{G(t)}\,\mathrm{d}t,$$

these new parameters mesure the arc lengths along the coordinate curves. Consider the First Fundamental Form $\widetilde{I}$ for these new coordinates: Clearly, $\widetilde{E} = \widetilde{G} = 1$. What can we say about $\widetilde{F}$?

Since $\det \widetilde{I}$ is the area of the parallelogram spanned by $\widetilde{\boldsymbol{x}}_u$ and $\widetilde{\boldsymbol{x}}_v$, we see that

$$\det \widetilde{I} = 1 \cdot 1 \cdot \sin\theta$$

(as $\widetilde{\boldsymbol{x}}_u$ and $\widetilde{\boldsymbol{x}}_v$ are of unit length by construction). On the other hand,

$$\det \widetilde{I} = \sqrt{\widetilde{E}\widetilde{G} - \widetilde{G}^2} = \sqrt{1 \cdot 1 - \widetilde{F}^2},$$

from which we conclude that $\widetilde{F} = \cos\theta$, as required.

(b) As we know from Gauss's Theorema Egregium, K only depends on E, F, G and their derivatives. We can compute K following the same procedure as in the proof of Theorema Egregium. Moreover, you can find (almost) this computation in the Solutions for homework question 15.2 (it does the case for the function $\theta(u, v) = uv$, but it is not a difficult exercises to update the solution for the case of the general function $\theta = \theta(u, v)$).

□

Proposition 14.11 (Lemma 3 in Do Carmo)**.** *Let $V \subset S$ be a coordinate neighbourhood of S such that the coordinate curves are asymptotic curves in V. Then the area A of any quadrilateral formed by the coordinate curves is smaller than 2π.*

Proof. By Proposition 14.9 we can choose the parametrisation so that the coordinate curves form a Chebyshev net. So, by Proposition 14.10(a), we can reparametrise it so, that $E = G = 1$ and $F = \cos\theta$.

Let R be a quadrilateral formed by coordinate curves (see Fig. 1). Then

$$A = \int_R \mathrm{d}A = \int_R \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v = \int_R \sqrt{1 - \cos^2\theta}\,\mathrm{d}u\,\mathrm{d}v = \int_R \sin\theta\,\mathrm{d}u\,\mathrm{d}v.$$

As $K = -1$ and also $K = \frac{\theta_{uv}}{\sin\theta}$ by Proposition 14.10(b), we conclude that

$$A = \int_R \theta_{uv}\,\mathrm{d}u\,\mathrm{d}v = \theta(u_1, v_1) - \theta(u_2, v_1) + \theta(u_2, v_2) - \theta(u_1, v_2) = \alpha_1 + \alpha_3 - (\pi - \alpha_2) - (\pi - \alpha_4) = \sum_{i=1}^{4} \alpha_i - 2\pi < 2\pi.$$

□

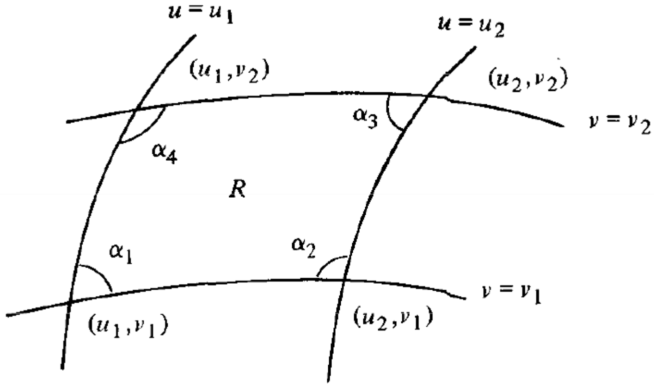

Fig.1 (copied from Do Carmo, Figure 5-54).

14.3 Step 3: Finishing the proof

So far we discussed the parametrisation only locally, but one can define the global extension $x : \mathbb{R}^2 \to S$ of it as follows (see Fig. 2):

- Take a point $p \in S$.
- Consider asymptotic curves α_1 and α_2 through p.
- For each $(s,t) \in \mathbb{R}^2$ lay of on α_1 the distance s, get a point p_1.
- Consider another asymptotic curve through p_1.
- lay of the distance t along that asymptotic curve to get $\boldsymbol{x}(s,t)$.

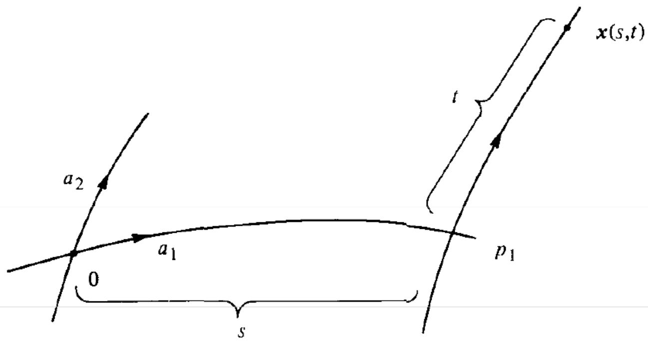

Fig.2 (copied from Do Carmo, Figure 5-55).

Then one can check (but it requires some work) that:

- $\boldsymbol{x}(s,t)$ is well defined for every $(s,t) \in \mathbb{R}^2$. (Here, one needs to use completeness of $\mathbb{H}^2$).
- For a fixed t, the curve $\boldsymbol{x}(s,t)$ is an asymptotic curve with s being it arc length. (Lemma 4 in Do Carmo).
- $\boldsymbol{x}(s,t)$ is a diffeomorphism, i.e. a local diffeomorphism and bijective map (Lemmas 5-8 in Do Carmo).

Now, we can consider the images by $\boldsymbol{x}$ of the nested squares with vertices $(\pm a, \pm a)$, $a \in \mathbb{Z}_+$. Their union cover S but each of them has finite area smaller that 2π. This contradicts to Lemma 1 saying that the area of hyperbolic plane is infinite.

14.4 Concluding remarks

Hilbert's Theorem actually states a bit more:

Theorem 14.12 (Hilbet's Theorem). *A complete geometric surface S with constant negative curvature cannot be isometrically immersed in $\mathbb{R}^3$.*

Here, "geometric surface" is a smooth glueing of open sets from R^2 with the first fundamental form on it (in the same way as hyperbolic plane is the upper half-plane with the first fundamental form).

If you are staying in Durham for one more year, you can see more on that in Riemannian Geometry IV course.