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Exercise on involute and evolute

Exercise: Let $\boldsymbol{\beta}(u) : I \to \mathbb{R}^2$ be a smooth regular curve. Suppose that $\kappa(u) \neq 0$ on I. Consider

$$\boldsymbol{\gamma}(u) = \boldsymbol{\beta}(u) - l_\beta(u)\boldsymbol{t}_\beta(u).$$

Show that $\beta(\mathbf{u})$ is the evolute of $\gamma(\mathbf{u})$ (and hence, that $\boldsymbol{\gamma}(u)$ is an involute of $\beta(\mathbf{u})$.

Solution: We need to show that

$$\boldsymbol{\beta}(u) = \boldsymbol{\gamma}(u) - \frac{1}{\kappa_\gamma(u)}\boldsymbol{n}_\gamma(u).$$

By definition of $\boldsymbol{\gamma}(u)$ this means that we want to show

$$\boldsymbol{\beta}(u) = (\boldsymbol{\beta}(u) - l_\beta(u)\boldsymbol{t}_\beta(u)) - \frac{1}{\kappa_\gamma(u)}\boldsymbol{n}_\gamma(u).$$

So, it is sufficient to show that

$$0 = -l_\beta(u)\boldsymbol{t}'_\beta(u) - \frac{1}{\kappa_\gamma(u)}\boldsymbol{n}_\gamma(u).$$

For simplicity we will assume that $\kappa_\beta(u) > 0$ (the case of $\kappa_\beta(u) < 0$ is very similar). Then we will show that

(1) $\boldsymbol{n}_\gamma(u) = \boldsymbol{t}_\beta(u)$ and

(2) $\kappa_\gamma(u) = -\frac{1}{l_\beta(u)}$

(check that if $\kappa_\beta(u) < 0$, then $\boldsymbol{n}_\gamma(u) = -\boldsymbol{t}_\beta(u)$ and $\kappa_\gamma(u) = \frac{1}{l_\beta(u)}$).

Also, for simplicity we assume that u is the unit length parameter for $\boldsymbol{\beta}$ (this assumption does not change the definition $\boldsymbol{\gamma}$ as no of its elements depends of the parametrisation (indeed, the point $\boldsymbol{\beta}(u)$, the length of a piece the the curve $l_\beta(u)$ and the unit tangent vector $\boldsymbol{t}_\beta(u)$ does not depend of the parametrisation).

Step 1. To show $\boldsymbol{n}_\gamma(u) = \boldsymbol{t}_\beta(u)$, compute $\boldsymbol{\gamma}'(u)$:

$$\boldsymbol{\gamma}'(u) \overset{\text{def}\,\gamma}{=} \boldsymbol{\beta}'(u) - l'_\beta(u)\boldsymbol{t}_\beta(u) - l_\beta(u)\boldsymbol{t}'_\beta(u) \overset{l'_\beta(u)=1}{=} \boldsymbol{\beta}'(u) - \boldsymbol{\beta}'(u) - l_\beta(u)\boldsymbol{t}'_\beta(u) \overset{\boldsymbol{t}'_\beta(s)=\kappa_\beta(s)\boldsymbol{n}_\beta(s)}{=} -l_\beta(u)\kappa_\beta(u)\boldsymbol{n}_\beta(u).$$

This implies that $\boldsymbol{t}_\gamma(u)$ is parallel to $\boldsymbol{n}_\beta(u)$, and hence, we conclude that $\boldsymbol{n}_\gamma(u)$ is parallel to $\boldsymbol{t}_\beta(u)$ (as the normal vector is orthogonal to the corresponding tangent). Since both $\boldsymbol{n}_\gamma(u)$ and $\boldsymbol{t}_\beta(u)$ are unit vectors we conclude that $\boldsymbol{n}_\gamma(u) = \pm\boldsymbol{t}_\beta(u)$. Finally, taking in account that $\boldsymbol{t}_\gamma(u) = -C(u)\boldsymbol{n}_\beta(u)$ with positive $C(u) = l_\beta\kappa_\beta(u)$, by rotating n_β clockwise and $\boldsymbol{t}_\gamma$ anti-clockwise, we obtain that $\boldsymbol{n}_\gamma(u) = \boldsymbol{t}_\beta(u)$

Step 2. Now, we need to prove $\kappa_\gamma(u) = -\frac{1}{l_\beta(u)}$. To compute $\kappa_\gamma(u)$, we need to take in account that u is not the unit speed parameter for γ. Denote $\hat{s}$ the unit speed parameter for γ.

From Lecture 7, formula (*) we have $\frac{d}{du} = \frac{d\hat{s}}{du}\frac{d}{d\hat{s}} = ||\boldsymbol{\gamma}'(u)||\frac{d}{d\hat{s}}$. Hence,

$$\boldsymbol{t}'_\gamma(u) \overset{(*)}{=} ||\boldsymbol{\gamma}'(u)||\frac{d}{d\hat{s}}\boldsymbol{t}_\gamma(u(\hat{s})) = ||\boldsymbol{\gamma}'(u)||(\boldsymbol{t}_\gamma(\hat{s}))' \overset{\boldsymbol{t}'_\gamma(\hat{s})=\kappa_\gamma(\hat{s})\boldsymbol{n}_\gamma(\hat{s})}{=} ||\boldsymbol{\gamma}'(u)||\kappa_\gamma(\hat{s})\boldsymbol{n}_\gamma(\hat{s}).$$

At the same time, by Step 1 we have $||\boldsymbol{\gamma}'(u)|| = l_\beta(u)\kappa_\beta(u)$, so we conclude

$$\boldsymbol{t}'_\gamma(u) = l_\beta(u)\kappa_\beta(u)\kappa_\gamma(u)\boldsymbol{n}_\gamma(u).$$

Since by Step 1 $\boldsymbol{t}_\gamma(u) = -\boldsymbol{n}_\beta(u)$, we have $\boldsymbol{t}'_\gamma(u) = -(n_\beta(u))' = -\kappa_\beta(u)\boldsymbol{t}_\beta(u)$ and therefore

$$-\kappa_\beta(u)\boldsymbol{t}_\beta(u) = l_\beta(u)\kappa_\beta(u)\kappa_\gamma(u)\boldsymbol{n}_\gamma(u).$$

Reducing by $\kappa_\beta(u)$ and taking in account that $\boldsymbol{n}_\gamma(u) = \boldsymbol{t}_\beta(u)$ in view of Stop 1, we get $-1 = l_\beta(u)\kappa_\beta(u)$ as required.

Remark. As you can see, the computation is not short, so it is not really supposed that you can complete it without help! (and definitely there is no reason to memorise...)