**Outline**[1]

# 2   Regular curves in $\mathbb{R}^n$

**Definition 2.1.**

(a) A *smooth curve* in $\mathbb{R}^n$ is a smooth (that is, infinitely differentiable) map

$$\boldsymbol{\alpha} \colon I \to \mathbb{R}^n,$$

then where $I$ is an open interval of $\mathbb{R}$ (so $I$ could be $(a, b)$ or $(-\infty, b)$ or $(a, +\infty)$ or $\mathbb{R}$).

(b) The *image*, $\boldsymbol{\alpha}(I)$, of $I$ under $\boldsymbol{\alpha}$ is called the *trace* of $\boldsymbol{\alpha}$. The variable $u \in I$ is called the *parameter* of $\boldsymbol{\alpha}$.

(c) If we write
$$\boldsymbol{\alpha}(u) = (\alpha_1(u), \alpha_2(u), \ldots, \alpha_n(u))$$

then each $\alpha_i \colon I \to \mathbb{R}$ is smooth. The vector

$$\boldsymbol{\alpha}'(u) = (\alpha_1'(u), \alpha_2'(u), \ldots, \alpha_n'(u))$$

is the *tangent vector* to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u)$.

(d) The curve $\boldsymbol{\alpha}$ is *regular* if $\boldsymbol{\alpha}'(u) \neq \mathbf{0} = (0, \ldots, 0)$ for all $u \in I$. The curve $\boldsymbol{\alpha}$ is *singular* at $\boldsymbol{\alpha}(u)$ if $\boldsymbol{\alpha}'(u) = \mathbf{0}$.

(e) If $\boldsymbol{\alpha}$ is a regular curve, we define the unit tangent vector

$$\boldsymbol{t}(u) = \frac{\boldsymbol{\alpha}'(u)}{\|\boldsymbol{\alpha}'(u)\|}.$$

If we want to stress that $\boldsymbol{t}$ is the unit tangent vector of the curve $\boldsymbol{\alpha}$, we also write $\boldsymbol{t_\alpha}$.

(f) If $\|\boldsymbol{\alpha}'(u)\| = 1$ for all $u \in I$ then $\boldsymbol{\alpha}$ is called *unit speed*.

**Example 2.2.**

(a) *The unit circle.* $\boldsymbol{\alpha} \colon \mathbb{R} \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (\cos u, \sin u)$. $\boldsymbol{\alpha}$ is smooth and unit speed.

(b) *The helix.* $\boldsymbol{\alpha} \colon \mathbb{R} \to \mathbb{R}^3$, $\boldsymbol{\alpha}(u) = (\cos u, \sin u, u)$. $\boldsymbol{\alpha}$ is smooth and regular.

(c) *The cusp.* $\boldsymbol{\alpha} \colon \mathbb{R} \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha} = (u^3, u^2)$ so $\boldsymbol{\alpha}$ is smooth. But $\boldsymbol{\alpha}'(s) = (3u^2, 2u)$, so $\boldsymbol{\alpha}'(0) = (0, 0)$.

(d) *The node.* $\boldsymbol{\alpha} \colon \mathbb{R} \to \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (u^3 - u, u^2 - 1)$. $\boldsymbol{\alpha}$ is smooth and regular but not injective, since $\boldsymbol{\alpha}(-1) = \boldsymbol{\alpha}(1)$.

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[1]This is an updated version of notes by Pavel Tumarkin, which was in its turn based on the notes by Olaf Post

**Definition 2.3.** Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^n$ be a smooth and regular curve. A *change of parameter* for $\boldsymbol{\alpha}$ is a function $h\colon J \longrightarrow I$ where $J$ is an open interval of $\mathbb{R}$ satisfying

  (a) $h$ is smooth;

  (b) $h'(t) \neq 0$ for all $t \in J$;

  (c) $h(J) = I$.

**Remark.** $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \circ h\colon J \longrightarrow \mathbb{R}^n$ is a smooth curve with the same trace as $\boldsymbol{\alpha}$.

**Example 2.4.** In the Example 2.2(a) take $J = \mathbb{R}$, $h(v) = 2v$. Then

$$\tilde{\boldsymbol{\alpha}}(v) = (\boldsymbol{\alpha} \circ h)(v) = \boldsymbol{\alpha}(2v) = (\cos 2v, \sin 2v).$$

**Definition 2.5.** The *arc length* of a curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^n$, measured from a point $\boldsymbol{\alpha}(u_0)$ for some $u_0 \in I$, is

$$\ell(u) := \int_{u_0}^{u} \|\boldsymbol{\alpha}'(v)\| \, dv.$$

**Remark.** If $\boldsymbol{\alpha}$ is unit speed ($\|\boldsymbol{\alpha}'(u)\| = 1$), then

$$\ell(u) = \int_{u_0}^{u} \|\boldsymbol{\alpha}'(s)\| \, ds = u - u_0.$$

So the parameter $u$ measures the arc length (up to an additive constant) and is called *arc length parameter*, $\boldsymbol{\alpha}$ is *parametrized by arc length*.

**Proposition 2.6.** Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^n$ be a smooth and regular curve. Choose $u_o \in I$, and let $\ell\colon I \longrightarrow \mathbb{R}$ be the arc length of $\boldsymbol{\alpha}$ w.r. to $u_0$. Define $J = \ell(I)$. Then $\ell^{-1}$ is a parameter change, and

$$\boldsymbol{\beta} = \boldsymbol{\alpha} \circ \ell^{-1}\colon J \longrightarrow \mathbb{R}^n$$

is parametrized by arc length.

**Example 2.7.** *The catenary.*

$$\boldsymbol{\alpha}\colon \mathbb{R} \longrightarrow \mathbb{R}^2, \quad \boldsymbol{\alpha}(u) = (u, \cosh u) \quad \Rightarrow \boldsymbol{\alpha}'(u) = (1, \sinh u)$$

$\boldsymbol{\alpha}$ is regular, $\|\boldsymbol{\alpha}'(u)\| = \sqrt{1 + \sinh^2 u} = \cosh u$,

$$s = \ell(u) = \int_{0}^{u} \|\boldsymbol{\alpha}'(t)\| \, dt = \int_{0}^{u} \cosh t \, dt = \sinh u$$

where we fixed $u_0 = 0$, and thus $u = \ell^{-1}(s) = \sinh^{-1} s$. So the arc-length parametrization of the catenary is

$$\boldsymbol{\beta} = \boldsymbol{\alpha}(\ell^{-1}(s)) = \left( \ln(s + \sqrt{s^2 + 1}), \cosh(\ln(s + \sqrt{s^2 + 1})) \right).$$

# 3 Plane curves

## 3.1 Tangent and normal vectors. Curvature

Let $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^2$ be a plane curve parametrized by arc length, i.e., $\boldsymbol{\alpha}'(s) = \mathbf{t}(s)$ is a unit vector.

**Definition 3.1.** The *unit normal vector* $\mathbf{n}(s)$ is the vector obtained by rotating $\mathbf{t}(s)$ anticlockwise through $\pi/2$.

In coordinates, if $\boldsymbol{\alpha}(s) = (x(s), y(s))$, then

$$\mathbf{t}(s) = (x'(s), y'(s)), \qquad \mathbf{n}(s) = (-y'(s), x'(s))$$

**Remark.** Differentiating the equation $1 = \|\mathbf{t}(s)\|^2 = \mathbf{t}(s) \cdot \mathbf{t}(s)$ gives

$$0 = \mathbf{t}'(s) \cdot \mathbf{t}(s) + \mathbf{t}(s) \cdot \mathbf{t}'(s) = 2\mathbf{t}'(s) \cdot \mathbf{t}(s).$$

In particular, $\mathbf{t}(s)$ and $\mathbf{t}'(s)$ are orthogonal, and hence $\mathbf{t}'(s)$ is parallel to the normal vector $\mathbf{n}(s)$ (which is also orthogonal to $\mathbf{t}(s)$). (Note that we use here the fact that we are in $\mathbb{R}^2$, otherwise the last conclusion that $\mathbf{t}'(s)$ is parallel to $\mathbf{n}(s)$ is not true!)

**Definition 3.2.** The *(signed) curvature* $\kappa(s)$ of a plane curve $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^2$ is defined by $\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s)$.

**Remark.** A way to compute: $\mathbf{n}(s) \cdot \mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) \cdot \mathbf{n}(s) = \kappa(s)$ (since $\mathbf{n}(s)$ is a unit vector), so we have

$$\kappa(s) = \mathbf{n}(s) \cdot \mathbf{t}'(s)$$

If $\boldsymbol{\alpha}$ is given by $\boldsymbol{\alpha}(s) = (x(s), y(s))$, where $s$ is the arc length, then

$$\kappa(s) = -y'(s)x''(s) + x'(s)y''(s),$$

provided the curve is parametrized by arc length.

**Example 3.3.**  (a) *Lines.* $\kappa(s) \equiv 0$.

(b) *Circles.* $\kappa(s) \equiv 1/r$ for a circle of radius $r$.

**Proposition 3.4.** Let $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (x(u), y(u))$, be a regular curve (not necessarily parametrized by arc length). Then

$$\kappa = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}},$$

where we omitted the argument $u$ of the functions $\kappa$, $x'$, $x''$, $y'$ and $y''$.

**Example.** *The ellipse.* Let $\boldsymbol{\alpha} \colon \mathbb{R} \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (a\cos u, b\sin u)$ for some constants $a, b > 0$. The curve is regular,

$$\kappa(u) = \frac{ab}{(a^2 \sin^2 u + b^2 \cos^2 u)^{3/2}}.$$

In particular, the curvature is always positive ($\kappa(u) > 0$ for all $u \in \mathbb{R}$), but not constant if $a \neq b$.

**Definition 3.5.** Let $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^2$ be a plane regular curve.

(a) A point $\boldsymbol{\alpha}(u_0)$ is an *inflection point* of $\boldsymbol{\alpha}$ if $\kappa(u) = 0$.

(b) A point $\boldsymbol{\alpha}(u_0)$ is a *vertex* of $\boldsymbol{\alpha}$ if $\kappa'(u) = 0$.

**Remark.** A vertex is well-defined, i.e. the definition does not depend on the parameter.

**Example 3.6.**   (a) *The cubic.* $\alpha(u) = (u, u^3)$. The only inflection point is $\boldsymbol{\alpha}(0) = (0, 0)$, there are no vertices.

(b) *The parabola.* $\boldsymbol{\alpha}(u) = (u, u^2)$. There are no inflection points, the only vertex is at $u = 0$.

(c) *The ellipse.* There are no inflection points, 4 vertices at $u = k\pi/2$.

**Theorem 3.7** (The 4-vertex theorem). Any simple smooth regular closed curve has at least 4 vertices.

Here *simple* means the curve has no self-intersections.

**Theorem 3.8** (The fundamental theorem of local theory of plane curves). Given a smooth function $\kappa \colon I \longrightarrow \mathbb{R}$, $s_0 \in I$, $a \in \mathbb{R}^2$ and a unit vector $v_0 \in \mathbb{R}^2$, there is a unique smooth regular curve $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^2$ parametrized by arc length with curvature $\kappa(s)$ and $\boldsymbol{\alpha}(s_0) = a$, $\boldsymbol{\alpha}'(s_0) = v_0$.

## 3.2   Evolute and involute of a plane curve

**Definition 3.9.** Let $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^2$ be a smooth regular curve parametrized by arc length.

(a) Suppose $\kappa(s) \neq 0$, then

$$\rho(s) = \frac{1}{|\kappa(s)|}$$

is called the *radius of curvature*. The point

$$\boldsymbol{e}(s) = \boldsymbol{\alpha}(s) + \frac{1}{\kappa(s)} \boldsymbol{n}(s)$$

is called the *center of curvature*. Here, $\boldsymbol{n}$ is the unit normal of $\boldsymbol{\alpha}$.

**Remark. Properties of the evolute.**
   $\boldsymbol{\alpha}$, $\boldsymbol{n}$ and $\kappa$ are smooth, so $\boldsymbol{e}$ is a smooth curve (whenever $\kappa(s) \neq 0$). Moreover,

$$\boldsymbol{e}'(s) = \boldsymbol{\alpha}'(s) + \frac{1}{\kappa(s)} \boldsymbol{n}'(s) - \frac{\kappa'(s)}{\kappa(s)^2} \boldsymbol{n}(s),$$

which implies

$$\boldsymbol{e}'(s) = -\frac{\kappa'(s)}{\kappa(s)^2} \boldsymbol{n}(s).$$

In particular, we have the following conclutions:

(a) $\boldsymbol{e}'(s)$ is *parallel* to the normal vector $\boldsymbol{n}(s)$ of the original curve $\boldsymbol{\alpha}$.

(b) $\boldsymbol{e}'(s) = \boldsymbol{0}$ iff $\kappa'(s) = 0$, i.e., the evolute is *singular* at $\boldsymbol{e}(s_0)$ iff $\boldsymbol{\alpha}(s_0)$ is a *vertex*.

(c) The parameter $s$ is *not* an arc length parameter of the evolute $\boldsymbol{e}$: $\|\boldsymbol{e}'(s)\| = |\frac{\kappa'(s)}{\kappa(s)^2}|$ which is not necessarily 1.

**Example 3.10.**   (a) *The ellipse.* $\boldsymbol{\alpha}(u) = (a\cos u, b\sin u)$ for $a > 0$, $b > 0$ and $a \neq b$.

$$\boldsymbol{e}(u) = (a\cos u, b\sin u) + \frac{a^2 \sin^2 u + b^2 \cos^2 u}{ab}(-b\cos u, -a\sin u).$$

(b) *The circle.* $\boldsymbol{e}(u) = $ the center.

# 4 Space curves (curves in $\mathbb{R}^3$)

## 4.1 The Serret – Frenet formulae

Let $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$ parametrized by arc length (i.e., $\boldsymbol{t} = \boldsymbol{\alpha}'$ is the unit tangent vector).

**Definition 4.1.** The *curvature* $\kappa \colon I \longrightarrow [0, \infty)$ of a space curve $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^3$ is defined by

$$\kappa(s) := \|\boldsymbol{t}'(s)\|.$$

**Remark.** The curvature of a *space* curve is always non-negative ($\kappa(s) \geq 0$). For *plane* curves, we introduced the *signed* curvature, which can have negative values. We will see the relation between both concepts later on.

**Definition 4.2.** Assume that $\kappa(s) > 0$. We define the *principal normal vector* $\boldsymbol{n}(s)$ by

$$\boldsymbol{n}(s) := \frac{1}{\kappa(s)} \boldsymbol{t}'(s).$$

Note that $\boldsymbol{n}(s)$ is really a *unit* vector (and also orthogonal to $\boldsymbol{t}(s)$). We have

$$\boldsymbol{t}'(s) = \kappa(s) \boldsymbol{n}(s).$$

**Remark. The *vector product* (or *cross-product*) $\times \colon \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ in $\mathbb{R}^3$.** Recall some facts about the vector product in $\mathbb{R}^3$. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$.

(a) The *vector product* is defined by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

(b) $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$, e.g., $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a} = 0$.

(c) Antisymmetry: $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$ (in particular, $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$).

(d) If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal unit vectors, then $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b})$ form an orthonormal basis, which is *positively* oriented. Moreover, one has

$$\boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}, \qquad (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{a} = \boldsymbol{b}$$

**Definition 4.3.** The vector $\boldsymbol{b} := \boldsymbol{t} \times \boldsymbol{n}$ is called the *binormal vector* of $\boldsymbol{\alpha}$, and $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ form an orthonormal basis called also *orthonormal frame*.

Since $\boldsymbol{b}'$ is orthogonal to $\boldsymbol{b}$ *and* to $\boldsymbol{t}$, $\boldsymbol{b}'$ is *parallel* to $\boldsymbol{n}$. In particular, the following definition makes sense:

**Definition 4.4.** The *torsion* $\tau \colon I \longrightarrow \mathbb{R}$ of the space curve $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^3$ is defined by

$$\boldsymbol{b}'(s) = \tau(s) \boldsymbol{n}(s).$$

**Remark.** Note that the torsion can have positive or negative values. Moreover, in some books, you will find the equation $\boldsymbol{b}' = -\tau \boldsymbol{n}$ as a definition of the torsion.

**Proposition 4.5** (*Serret-Frenet equations*)**.** Let $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^3$ be a space curve parametrized by arc length with unit tangent, normal and binormal vectors $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$. Then

$$\boldsymbol{t}' = \kappa \boldsymbol{n} \tag{4.2}$$

$$\boldsymbol{n}' = -\kappa \boldsymbol{t} - \tau \boldsymbol{b} \tag{4.6}$$

$$\boldsymbol{b}' = \tau \boldsymbol{n} \tag{4.5}$$

or in matrix form

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}.$$

Let us now show how to calculate the torsion and curvature for a space curve which is not necessarily parametrized by arc length. This is of practical relevance, since a parametrization is in general not unit speed (i.e., the parameter is not arc length).

**Theorem 4.6.** Let $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^3$ be a regular space curve, not necessarily parametrized by arc length. Then the curvature and torsion of $\boldsymbol{\alpha}$ are given by

$$\kappa = \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3} \quad \text{and} \quad \tau = -\frac{(\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'') \cdot \boldsymbol{\alpha}'''}{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|^2}$$

(as functions of $u$), respectively.

**Example 4.7.** *The helix.* Let $\boldsymbol{\alpha} \colon \mathbb{R} \longrightarrow \mathbb{R}^3$ be given by $\boldsymbol{\alpha}(u) = (a \cos u, a \sin u, u)$ for $a > 0$ (this is a particular case of a helix, see Exercise 4.5). Then $\kappa = \dfrac{a}{a^2 + 1}$, $\tau(u) = -\dfrac{1}{a^2 + 1}$.

**Remark** (Geometric meaning of torsion)**.** The plane through $\boldsymbol{\alpha}(s)$ spanned by $\boldsymbol{t}(s)$ and $\boldsymbol{n}(s)$ is called the *osculating plane*.

The torsion of a curve measures the rate at which the curve pulls away from the osculating plane.

**Proposition 4.8.** Let $\boldsymbol{\alpha} \colon I \to \mathbb{R}^3$ be a smooth curve, $\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'' \neq \boldsymbol{0}$ for $u \in I$. Then $\tau(u) \equiv 0$ if there is a plane $\Pi \subset \mathbb{R}^3$ containing $\boldsymbol{\alpha}(I)$.

We can now express one of the main results on space curve (similar to Theorem 3.8):

**Theorem 4.9** (The fundamental theorem of local theory of space curves)**.** Given smooth functions $\kappa \colon I \longrightarrow (0, \infty)$ and $\tau \colon I \longrightarrow \mathbb{R}$, there exists a smooth regular curve $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^3$ parametrized by arc length such that $\kappa$ and $\tau$ are the curvature and torsion of $\boldsymbol{\alpha}$. Moreover, $\boldsymbol{\alpha}$ is unique up to translations (of the *starting point*) and rotation (of the *starting orthonormal basis*).

**Remark 4.10.** *Local canonical form of a space curve.* Let $\boldsymbol{\alpha} \colon I \longrightarrow \mathbb{R}^3$ be a space curve parametrized by arc length with $0 \in I$. Then

$$\boldsymbol{\alpha}(s) = \boldsymbol{\alpha}(0) + s \boldsymbol{\alpha}'(0) + \frac{s^2}{2!} \boldsymbol{\alpha}''(0) + \frac{s^3}{3!} \boldsymbol{\alpha}'''(0) + O(s^4)$$

$$= \boldsymbol{\alpha}(0) + s \boldsymbol{t}(0) + \frac{s^2}{2!} \underbrace{\boldsymbol{t}'(0)}_{=\kappa(0)\boldsymbol{n}(0)} + \frac{s^3}{3!} \underbrace{\boldsymbol{t}''(0)}_{=\kappa'(0)\boldsymbol{n}(0)+\kappa(0)(-\kappa(0)\boldsymbol{t}(0)-\tau(0)\boldsymbol{b}(0))} + O(s^4)$$

by the Serret-Frenet formulae. In paricular,

$$\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(0) = \left(s - \frac{\kappa(0)^2 s^3}{6}\right) \boldsymbol{t}(0) + \left(\frac{\kappa(0) s^2}{2} + \frac{\kappa'(0) s^3}{6}\right) \boldsymbol{n}(0) - \frac{\kappa(0)\tau(0) s^3}{6} \boldsymbol{b}(0) + O(s^4).$$

If we choose the coordinate system such that $\boldsymbol{t}(0) = (1, 0, 0)$, $\boldsymbol{n}(0) = (0, 1, 0)$ and $\boldsymbol{b}(0) = (0, 0, 1)$, and if we write $\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(0) = (x(s), y(s), z(s))$, then

$$x(s) = s - \frac{\kappa(0)^2 s^3}{6}$$
$$y(s) = \frac{\kappa(0) s^2}{2} + \frac{\kappa'(0) s^3}{6}$$
$$z(s) = -\frac{\kappa(0) \tau(0) s^3}{6}.$$

These equations are called the *local canonical form* of a space curve $\boldsymbol{\alpha}$.

# 5  A bit of Analysis (should have been a reminder)

We consider the Euclidean space

$$\mathbb{R}^n = \big\{ \boldsymbol{x} = (x_1, \ldots, x_n) \,\big|\, x_i \in \mathbb{R}, i = 1, \ldots, n \big\}$$

**Definition 5.1.**

(a) A *ball of radius $r > 0$ with center $\boldsymbol{a} \in \mathbb{R}^n$* in $\mathbb{R}^n$ is defined by

$$B_r(\boldsymbol{a}) := \big\{ \boldsymbol{x} \in \mathbb{R}^n \,\big|\, \|\boldsymbol{x} - \boldsymbol{a}\| = \sqrt{(x_1 - a_1)^2 + \ldots + (x_n - a_n)^2} < r \big\}.$$

(b) A subset $U \subset \mathbb{R}^n$ is called *open*, if for any $\boldsymbol{y} \in U$ there exists $r > 0$ such that $B_r(\boldsymbol{y}) \subset U$, i.e.

$$\forall\, \boldsymbol{y} \in U \;\exists\, r > 0 : \quad B_r(\boldsymbol{y}) \subset U.$$

**Example 5.2.**

(a) Interval $(a, b) \subset \mathbb{R}$ is open.

(b) Closed interval $[a, b] \subset \mathbb{R}$ is not open.

(c) The ball $B_r(\boldsymbol{a})$ is an open subset of $\mathbb{R}^n$ for any $\boldsymbol{a} \in \mathbb{R}^n$ and $r > 0$.

(d) The *(open) cube* $(a_1, b_1) \times \ldots \times (a_n, b_n)$ is an open subset for any $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$. Note that for $n = 1$, a cube is an interval, and for $n = 2$, a cube is a rectangle (without the boundary).

(e) The entire space $\mathbb{R}^n$ and the empty set $\emptyset$ are open.

Now let $U \subset \mathbb{R}^n$ be open, $\boldsymbol{f} : U \longrightarrow \mathbb{R}^m$ be a map, i.e.,

$$\boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} f_1(u_1, \ldots, u_n) \\ \vdots \\ f_m(u_1, \ldots, u_n) \end{pmatrix}$$

for any $\boldsymbol{u} = (u_1, \ldots, u_n) \in U$. We say that $\boldsymbol{f}$ is *smooth* if the (scalar) functions $f_i : U \longrightarrow \mathbb{R}$ are smooth for all $i = 1, \ldots m$, i.e., if all partial derivatives of all order exist and are continuous.

**Example 5.3.**

(a) $\boldsymbol{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ ($U = \mathbb{R}^2$, $n = 2$, $m = 3$) with

$$\boldsymbol{f}(u_1, u_2) = \begin{pmatrix} u_1 \\ u_2 \\ u_1^2 + u_2^2 \end{pmatrix}$$

is a smooth map.

(b) $\boldsymbol{f}\colon B_1(\boldsymbol{0}) \longrightarrow \mathbb{R}^3$ ($U = B_1(\boldsymbol{0}) \subset \mathbb{R}^2$, $n = 2$, $m = 3$) with

$$\boldsymbol{f}(u_1, u_2) = \begin{pmatrix} u_1 \\ u_2 \\ \sqrt{1 - u_1^2 - u_2^2} \end{pmatrix}$$

is a smooth map as well.

For (scalar) functions, even of more than one variable, we know how to derive, e.g., if $f(x, y) = x^2 y + 3y^3$, then

$$\frac{\partial f}{\partial x}(x, y) = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = x^2 + 9y^2.$$

**Definition 5.4.** Let $U \subset \mathbb{R}^n$ be open, let $\boldsymbol{f}\colon U \longrightarrow \mathbb{R}^m$ be a smooth map and let $\boldsymbol{p} \in U$. The *Jacobi matrix* of $\boldsymbol{f}$ at $\boldsymbol{p}$ is the $(m \times n)$-matrix given by

$$J_{\boldsymbol{p}}\boldsymbol{f} := \begin{pmatrix} \partial_1 f_1(\boldsymbol{p}) & \cdots & \partial_n f_1(\boldsymbol{p}) \\ \vdots & & \vdots \\ \partial_1 f_m(\boldsymbol{p}) & \cdots & \partial_n f_m(\boldsymbol{p}) \end{pmatrix} \quad \text{where} \quad \partial_i f_j(\boldsymbol{p}) := \frac{\partial}{\partial u_i} f_j(u) \Big|_{u=p}, \quad i = 1, \ldots, n.$$

The *derivative* of $\boldsymbol{f}$ at $p$ is the linear map

$$\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad h \mapsto (\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(h) = J_{\boldsymbol{p}}\boldsymbol{f} \cdot h$$

Note that the Jacobi matrix is just the matrix representation of the derivative in the standard basis.

**Remark.** Since $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}$ is linear, its image (range) $(\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\mathbb{R}^n)$ is a vector subspace of $\mathbb{R}^m$, spanned by

$$\{(\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\boldsymbol{e}_1), \ldots, (\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\boldsymbol{e}_n)\},$$

where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the standard basis in $\mathbb{R}^n$. Observe that

$$(\partial_i \boldsymbol{f}(\boldsymbol{p}) :=)(\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\boldsymbol{e}_i) = \begin{pmatrix} \partial_i f_1(\boldsymbol{p}) \\ \vdots \\ \partial_i f_m(\boldsymbol{p}) \end{pmatrix}$$

which is just the $i^{\text{th}}$ column of the Jacobi matrix $J_{\boldsymbol{p}}\boldsymbol{f}$.

**Example 5.5.**

(a) $\boldsymbol{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\boldsymbol{f}(u, v) = \begin{pmatrix} u \\ v \\ u^2 + v^2 \end{pmatrix} \quad \text{then} \quad J_{(u,v)}\boldsymbol{f} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2u & 2v \end{pmatrix}.$$

At $\boldsymbol{p} = (0, 0)$, the image of $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}$ is spanned by $(1, 0, 0)$ and $(0, 1, 0)$.

(b) $\boldsymbol{f}: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$,

$$\boldsymbol{f}(u, v) = \begin{pmatrix} u \\ v^2 \\ uv \end{pmatrix} \quad \text{then} \quad J_{(u,v)}\boldsymbol{f} = \begin{pmatrix} 1 & 0 \\ 0 & 2v \\ v & u \end{pmatrix}.$$

At $\boldsymbol{p} = (0, 0)$, the image of $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}$ is spanned by $\{(1, 0, 0), (0, 0, 0)\}$, i.e., by $(1, 0, 0)$ (the $x$-axis).

(c) $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$$f(x, y, z) := 2x^2 + y^2 - z^2, \qquad J_{(x,y,z)}f = (4x, 2y, -2z)$$

(the *gradient* of $f$). Note that the Jacobi matrix of a scalar function is just the gradient. Here, the image of $\mathrm{d}_{\boldsymbol{p}}f$ is either $\mathbb{R}$ (if $(x, y, z) \neq \boldsymbol{0}$) or $\{0\}$ (if $(x, y, z) = \boldsymbol{0}$).

Let us finally motivate the *implicit function theorem*

**Example 5.6.** Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by $f(u, v) = u^2 + v^2$. We want to solve the equation

$$f(u, v) = c$$

near some point $(a, b) \in \mathbb{R}^2$ for $c := f(a, b) \geq 0$, i.e., we look for a function $g(u) = v$ such that $f(u, g(u)) = c$. The implicit function tells us that if $\partial_v f(u_0, v_0) \neq 0$ then this is possible. Here, $\partial_v f(a, b) = 2b$, and a simple calculation shows that

$$f(u, v) = c \iff v = \begin{cases} \sqrt{c - u^2}, & \text{if } b > 0, \\ -\sqrt{c - u^2}, & \text{if } b < 0. \end{cases}$$

**Theorem 5.7** (Implicit function theorem). Let $W \subset \mathbb{R}^p \times \mathbb{R}^m$ be open and $\boldsymbol{f}: W \longrightarrow \mathbb{R}^m$ be smooth. Let $(\boldsymbol{a}, \boldsymbol{b}) \in W$ ($\boldsymbol{a} \in \mathbb{R}^p$, $\boldsymbol{b} \in \mathbb{R}^m$) and $\boldsymbol{c} := f(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^m$. Consider a function $\boldsymbol{\varphi}: W \cap \mathbb{R}^m \to \mathbb{R}^m$ defined by $\boldsymbol{y} \mapsto \boldsymbol{f}(\boldsymbol{a}, \boldsymbol{y})$. Its Jacobi matrix is

$$J(\boldsymbol{a}, \boldsymbol{y}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{a}, \boldsymbol{y}) = \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1}(\boldsymbol{a}, \boldsymbol{y}) & \cdots & \dfrac{\partial f_1}{\partial y_m}(\boldsymbol{a}, \boldsymbol{y}) \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial y_1}(\boldsymbol{a}, \boldsymbol{y}) & \cdots & \dfrac{\partial f_m}{\partial y_m}(\boldsymbol{a}, \boldsymbol{y}) \end{pmatrix}$$

Assume that $J(\boldsymbol{a}, \boldsymbol{y})$ is invertible at $\boldsymbol{y} = \boldsymbol{b}$. Then there exist open sets $U \subset \mathbb{R}^p$, $\boldsymbol{a} \in U$, and $V \subset \mathbb{R}^m$, $\boldsymbol{b} \in V$, and a smooth map $\boldsymbol{g}: U \to V$ with $\boldsymbol{g}(\boldsymbol{a}) = \boldsymbol{b}$ such that

$$\{(\boldsymbol{x}, \boldsymbol{y}) \in U \times V \mid \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{c}\} = \{(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) \mid \boldsymbol{x} \in U\}$$

(i.e. the level set of points $(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{c}$ is locally a *graph* of some smooth function $\boldsymbol{g}: U \to V$).

We will use this theorem in a particular case of $m = 1$: having a function

$$f: \mathbb{R}^{p+1} \to \mathbb{R}, \quad (x_1, \ldots, x_p, y) \mapsto f(\boldsymbol{x}, y), \quad f(\boldsymbol{x}_0, y_0) = c$$

with $\dfrac{\partial f}{\partial y}(\boldsymbol{x}_0, y_0) \neq 0$, one has $y = g(\boldsymbol{x})$ in a neighborhood of $\boldsymbol{x}_0$ for $f(\boldsymbol{x}, y) = c$.

# 6 Surfaces

Recall that we defined a curve as a smooth map $\boldsymbol{\alpha}: I \longrightarrow \mathbb{R}^n$. So a curve is a deformation of an interval, i.e., a piece of the real line.

Similarly, we look to define a surface as a deformation of an open subset in $\mathbb{R}^2$. Intuitively, a surface in $\mathbb{R}^n$ ($n \geq 3$) is a subset of $\mathbb{R}^n$ that looks locally like a subset of $\mathbb{R}^2$.

## 6.1 Parametrizations of regular surfaces

**Definition 6.1.** A subset $S \subset \mathbb{R}^3$ is a *regular surface* if for every point $p \in S$ there exists an open set $V$ in $\mathbb{R}^3$ containing $p$ and a map $\boldsymbol{x} \colon U \longrightarrow S \cap V$, where $U$ is an open subset of $\mathbb{R}^2$, such that

(a) $\boldsymbol{x}$ is a smooth map; that is, if

$$\boldsymbol{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$$

then $x_1, x_2, x_3$ are smooth functions.

(b) $\boldsymbol{x} \colon U \longrightarrow S \cap V$ is a homeomorphism, that is, $\boldsymbol{x}$ has a continuous inverse $\boldsymbol{x}^{-1} \colon S \cap V \longrightarrow U$ *(this condition excludes self-intersections).*

(c) The partial derivatives $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are linearly independent for all $(u, v) \in U$ *(this condition excludes singularities and dimension reduction).*

$\boldsymbol{x}$ is called a *local parametrization* of $S$ at $p$, and $\boldsymbol{x}^{-1}$ is called a *local coordinate chart.*

Let us now come to some main classes of examples of surfaces:

## 6.2 Graphs of functions and level sets as surfaces

**Proposition 6.2.** Let $U \subset \mathbb{R}^2$ be open and $g \colon U \longrightarrow \mathbb{R}$ be a smooth function. Then the graph of $g$,

$$\operatorname{graph}(g) := \left\{ (u, v, g(u, v) \in \mathbb{R}^3 \,\middle|\, (u, v) \in U \right\}$$

is a regular surface in $\mathbb{R}^3$.

**Example 6.3.**

(a) Let $U = \mathbb{R}^2$ and

$$g(u, v) = \frac{u^2}{a^2} + \frac{v^2}{b^2},$$

then the graph of $g$ is a surface: an *elliptic paraboloid.*

(b) Similarly, let

$$g(u, v) = \frac{u^2}{a^2} - \frac{v^2}{b^2},$$

then the graph of $g$ is a *hyperbolic paraboloid.*

**Example 6.4.** The *sphere* of radius $r > 0$ and center $\boldsymbol{0}$ is defined as

$$S(r) := \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 - r^2 = 0 \right\}.$$

**Example 6.5.** Consider the function $f \colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2 + z^2$. Then the sphere $S(r)$ of radius $r > 0$ is the level set $r^2$ of $f$, i.e.,

$$S(r) = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, f(x, y, z) = r^2 \right\} =: f^{-1}(r^2)$$

All the level sets $f^{-1}(r^2)$ are *regular surfaces*, except for $c = r^2 = 0$. The value $c = 0$ corresponds to the point $\boldsymbol{x} = (x, y, z) = \boldsymbol{0}$. Note that

$$\nabla f = (\partial_x f, \partial_y f, \partial_z f) = (2x, 2y, 2z)$$

and that $\nabla f(\boldsymbol{x}) = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$. We have to exclude such values!

**Definition 6.6.** Let $U \subset \mathbb{R}^3$ be open and $f\colon U \longrightarrow \mathbb{R}$ be smooth. A value $c \in \mathbb{R}$ in the range $f(U)$ of $f$ is called *regular value* of $f$ if $\nabla f(\boldsymbol{p}) = (\partial_x f, \partial_y f, \partial_z f)(\boldsymbol{p}) \neq \boldsymbol{0}$ for all $\boldsymbol{p} \in U$ such that $f(\boldsymbol{p}) = c$.

A point $\boldsymbol{p}$ is called *critical point* if $\nabla f(\boldsymbol{p}) = \boldsymbol{0}$. In this case $c = f(\boldsymbol{p})$ is a *critical value* of $f$.

So $c = r^2 > 0$ is a regular value of $f$ from the previous example, and $c = 0$ is a critical value.

**Proposition 6.7.** Let $U \subset \mathbb{R}^3$ be open and $f\colon U \longrightarrow \mathbb{R}$ be smooth, let $c \in f(U)$ be a regular value of $f$. Then

$$f^{-1}(c) := \left\{\, \boldsymbol{x} \in U \,\middle|\, f(\boldsymbol{x}) = c \,\right\}$$

is a regular surface.

**Example 6.8.**

(a) $S(r) = \{\, (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2 \,\}$ is the level set of $f$, where $f(x, y, z) = x^2 + y^2 + z^2$, i.e., $S(r) = f^{-1}(r^2)$. $S(r)$ is a regular surface if $r > 0$.

(b) Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be given by $f(x, y, z) = x^2 + y^2 - z^2$. Let $S = f^{-1}(1)$ be the level set $1$ of $f$. Since $c = 1$ is a regular value of $f$, $S$ is a regular surface, a *hyperboloid of one sheet*.

(c) With the same $f$ as before, $f^{-1}(-1)$ is called the *hyperboloid of two sheets*. The value $-1$ is again a regular value, so the hyperboloid of two sheets is regular.

(d) A cylinder given by those points $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 = 1$ is a regular surface.

## 6.3 Change of parameters

**Definition 6.9.** Let $U, V$ be two open sets. A smooth map $\boldsymbol{h}\colon V \longrightarrow U$ is called a *diffeomorphism* if it is bijective and if the inverse $\boldsymbol{h}^{-1}\colon U \longrightarrow V$ is also smooth.

**Example 6.10.** Let $U = V = \mathbb{R}$. Then $\boldsymbol{h}(x) = x$ is a diffeomorphism, but $\boldsymbol{h}(x) = x^3$ is not.

**Proposition 6.11.**   (a) Let $S \subset \mathbb{R}^3$ be a surface and let $\boldsymbol{x}\colon U \subset \mathbb{R}^2 \longrightarrow S$ be a local parametrization. Let $\boldsymbol{h}\colon V \subset \mathbb{R}^2 \longrightarrow U$ be a diffeomorphism. Then $\boldsymbol{y} = \boldsymbol{x} \circ \boldsymbol{h}\colon V \longrightarrow S$ is also a local parametrization.

(b) Let $\boldsymbol{x}\colon U \longrightarrow S$ and $\boldsymbol{y}\colon V \longrightarrow S$ be two local parametrizations with $\boldsymbol{x}(U) = \boldsymbol{y}(V) \subset S$ (i.e., $\boldsymbol{x}$ and $\boldsymbol{y}$ cover the same region of the surface). Then $\boldsymbol{x}^{-1} \circ \boldsymbol{y}\colon V \longrightarrow U$ is a diffeomorphism.

## 6.4 Special surfaces

**Surfaces constructed by a plane and space curves.**

**Example 6.12. Surface of revolution.** Let $I$ be an open interval in $\mathbb{R}$ and $\tilde{\boldsymbol{\alpha}}\colon I \longrightarrow \mathbb{R}^2$ be a regular smooth plane curve, $\tilde{\boldsymbol{\alpha}}(v) = (f(v), g(v))$. Define a space curve $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$. Assume that $\boldsymbol{\alpha}$ has no self-intersections (i.e. $\boldsymbol{\alpha}(u) \neq \boldsymbol{\alpha}(v)$ if $u \neq v$) and that $f(v) \neq 0$, so $\boldsymbol{\alpha}$ does not meet the $z$-axis.

Now rotate $\boldsymbol{\alpha}$ about the $z$-axis. The set

$$S := \left\{\, (f(v) \cos u, f(v) \sin u, g(v)) \,\middle|\, u \in \mathbb{R}, v \in I \,\right\}$$

is a surface, called a *surface of revolution*.

The curve $\boldsymbol{\alpha}$ is called the *generating curve*. The circles swept out by points of $\boldsymbol{\alpha}$ are called *parallels*, and the curves obtained by rotating $\boldsymbol{\alpha}$ through a fixed angle are *meridians*.

Examples: cylinder ($\boldsymbol{\alpha}$ is a vertical line), *catenoid* ($\boldsymbol{\alpha}(v) = (\cosh v, 0, v)$, $v \in \mathbb{R}$).

**Example 6.13. Canal surfaces.**

Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a smooth regular non-self-intersecting space curve parametrized by arc length. Choose $r > 0$ small enough, and consider the family of circles in the normal plane (i.e., spanned by $\boldsymbol{n}(s)$ and $\boldsymbol{b}(s)$ with center $\boldsymbol{\alpha}(s)$ and radius $r$. These form a surface called a *canal surface* or *tubular neighbourhood* of $\boldsymbol{\alpha}$. This surface is parametrized by

$$\boldsymbol{x}(s, \vartheta) = \boldsymbol{\alpha}(s) + r(\boldsymbol{n}(s) \cos \vartheta + \boldsymbol{b}(s) \sin \vartheta).$$

**Example 6.14. Ruled surfaces.** Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a smooth regular space curve (without self-intersections) and $\boldsymbol{w}\colon I \longrightarrow \mathbb{R}^3$ be a smooth map which is never zero. Suppose that $\boldsymbol{\alpha}'(u)$ is not parallel to $\boldsymbol{w}(u)$ (where $\boldsymbol{w}(u)$ is viewed as a vector). We consider the family of segments of lines through $\boldsymbol{\alpha}(u)$ and parallel to $\boldsymbol{w}(u)$.

These form a surface call a *ruled surface*. If we take $J = (-a, a)$, with $a$ small enough, then

$$\boldsymbol{x}(u, v) = \boldsymbol{\alpha}(u) + v\boldsymbol{w}(u), u \in I, v \in J$$

is a parametrization of a ruled surface.

**Example 6.15.** $f(x, y, z) := x^2 + y^2 - z^2$ defines a smooth function $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$, and 1 is a regular value, hence $S = f^{-1} = \{\, (x, y, z) \,|\, x^2 + y^2 - z^2 = 1 \,\}$ is a regular surface, a *hyperboloid of one sheet*. It is a surface of revolution and a ruled surface.

# 7 Tangent plane, first fundamental form and area

## 7.1 The tangent plane

**Definition 7.1.** Let $S$ be a regular surface and $p \in S$. A *tangent vector* to $S$ at $p$ is the tangent vector $\alpha'(0) \in \mathbb{R}^3$ of a smooth (not necessarily regular) curve $\alpha\colon (-\varepsilon, \varepsilon) \longrightarrow S \subset \mathbb{R}^3$ with $\alpha(0) = p$ (for some $\varepsilon > 0$).

Let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization of $S$, $\boldsymbol{q} \in U$, $\boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{p}$. Recall that the differential (or derivative) $d_{\boldsymbol{q}}\boldsymbol{x}$ is a linear map $d_{\boldsymbol{q}}\boldsymbol{x}\colon \mathbb{R}^2 \to \mathbb{R}^3$. By the definition of a regular surface, $d_{\boldsymbol{q}}\boldsymbol{x}$ has full rank at every point, so the dimension of the image is equal to 2.

**Definition 7.2.** The plane $d_{\boldsymbol{q}}\boldsymbol{x}(\mathbb{R}^2)$ is called the *tangent plane* to $S$ at $\boldsymbol{p}$ and is denoted by $T_{\boldsymbol{p}}S$.

**Proposition 7.3.** Let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization of a regular surface $S$ with $U \subset \mathbb{R}^2$ open, and let $\boldsymbol{q} \in U$. Then Then the tangent plane $T_{\boldsymbol{p}}S$ coincides with the set of all tangent vectors to $S$ at $\boldsymbol{p}$.

**Remark 7.4.** (a) Since the definition of a tangent vector does not depend on a parametrization of $S$, Prop. 7.3 implies that the tangent plane does not depend on a parametrization either.

(b) If $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ and $\boldsymbol{w} = \alpha'(0)$, then $w$ has coordinates $(u'(0), v'(0))$ with respect to the basis $\{\boldsymbol{x}_u(\boldsymbol{q}), \boldsymbol{x}_v(\boldsymbol{q})\}$.

**Example 7.5.**

(a) **Tangent plane to graph of a function:** Let $g\colon U \longrightarrow \mathbb{R}$ be a smooth function on an open subset $U$ of $\mathbb{R}^2$, i.e.

$$S := \operatorname{graph} g = \{\, (u, v, g(u, v)) \,|\, (u, v) \in U \,\}$$

is a regular surface with parametrisation $\boldsymbol{x}(u, v) := (u, v, g(u, v))$. Then the tangent plane $T_{\boldsymbol{p}}S$ to $S$ at $\boldsymbol{p} = (u, v, g(u, v))$ is generated by

$$\{\boldsymbol{x}_u(\boldsymbol{q}), \boldsymbol{x}_v(\boldsymbol{q})\} = \{(1, 0, g_u(u, v)), (0, 1, g_v(u, v))\},$$

where $\boldsymbol{q} = (u, v)$.

(b) **Tangent plane to a level set of a function:** Let $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function, and let $c \in \mathbb{R}$ be a regular value of $f$ (i.e., $\nabla f(\boldsymbol{p}) \neq \boldsymbol{0}$ for all $\boldsymbol{p} \in \mathbb{R}^3$ with $f(\boldsymbol{p}) = c$). We have seen that $S := f^{-1}(c)$ is a regular surface.

**Lemma 7.6.** Let $\boldsymbol{p} \in S$, then $T_{\boldsymbol{p}}S$ is the plane in $\mathbb{R}^3$ *orthogonal* to $\nabla f(\boldsymbol{p})$.

## 7.2 The first fundamental form

Let $\boldsymbol{p} \in S$. We can consider the restriction of the inner product $(\cdot): \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$, $(\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{v} \cdot \boldsymbol{w}$, to $T_{\boldsymbol{p}}S \subset \mathbb{R}^3$. We denote the restriction by $\langle \cdot, \cdot \rangle_{\boldsymbol{p}}$, i.e.,

$$\langle \cdot, \cdot \rangle_{\boldsymbol{p}}: T_{\boldsymbol{p}}S \times T_{\boldsymbol{p}}S \longrightarrow \mathbb{R}, \qquad (\boldsymbol{w}_1, \boldsymbol{w}_2) \mapsto \boldsymbol{w}_1 \cdot \boldsymbol{w}_2.$$

This map is

- *bilinear*, i.e, linear in both of its arguments;

- *symmetric*, i.e., $\langle \boldsymbol{w}_2, \boldsymbol{w}_1 \rangle_{\boldsymbol{p}} = \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle_{\boldsymbol{p}}$ for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_{\boldsymbol{p}}S$;

- and *positive*, i.e., $\|\boldsymbol{w}\|_{\boldsymbol{p}}^2 := \langle \boldsymbol{w}, \boldsymbol{w} \rangle \geq 0$ and $\|\boldsymbol{w}\|_{\boldsymbol{p}}^2 = 0$ implies $\boldsymbol{w} = 0$ for all $\boldsymbol{w} \in T_{\boldsymbol{p}}S$.

We can now measure the length of a tangent vector $\boldsymbol{w} \in T_{\boldsymbol{p}}S$ and the angle between two tangent vectors $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_{\boldsymbol{p}}S$ by

$$\sqrt{\langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\boldsymbol{p}}} \qquad \text{and} \qquad \cos \vartheta = \frac{\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle_{\boldsymbol{p}}}{\sqrt{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle_{\boldsymbol{p}}} \sqrt{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle_{\boldsymbol{p}}}}.$$

A quadratic form $I_{\boldsymbol{p}}$ is obtained from a bilinear form $\langle \cdot, \cdot \rangle_{\boldsymbol{p}}$ by setting $I_{\boldsymbol{p}}(\boldsymbol{w}) := \langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\boldsymbol{p}}$.

**Definition 7.7.** The quadratic form $I_{\boldsymbol{p}}: T_{\boldsymbol{p}}S \longrightarrow \mathbb{R}$, $I_{\boldsymbol{p}}(\boldsymbol{w}) := \langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\boldsymbol{p}} = \|\boldsymbol{w}\|^2$ is called the *first fundamental form* at $\boldsymbol{p} \in S$.

**Definition 7.8.** The functions $E, F, G: U \longrightarrow \mathbb{R}$ defined by

$$E := \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle_{\boldsymbol{p}}, \quad F := \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle_{\boldsymbol{p}}, \quad G := \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle_{\boldsymbol{p}}$$

are called the *coefficients* of the first fundamental form in the local parametrization $\boldsymbol{x}: U \longrightarrow S$.

Note that the coefficients of the first fundamental form depend on the parametrisation $\boldsymbol{x}$!

**Remark 7.9.** If $(a, b) \in \mathbb{R}^2$ are the coordinates of a vector $\boldsymbol{w} \in T_{\boldsymbol{p}}S$ with respect to the basis $\{\boldsymbol{x}_u(\boldsymbol{q}), \boldsymbol{x}_v(\boldsymbol{q})\}$, then

$$I_{\boldsymbol{p}}(\boldsymbol{w}) = a^2 E + 2ab F + b^2 G = \begin{pmatrix} a & b \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$

Since $I_{\boldsymbol{p}}$ is positive ($I_{\boldsymbol{p}}(\boldsymbol{w}) = \|\boldsymbol{w}\|^2 \geq 0$ and $I_{\boldsymbol{p}}(\boldsymbol{w}) = 0$ implies $\boldsymbol{w} = \boldsymbol{0}$), we have

$$E > 0, \quad G > 0 \quad \text{and} \quad \det \begin{pmatrix} E & F \\ F & G \end{pmatrix} = EG - F^2 > 0.$$

**Example 7.10.** Let $S$ be a plane in $\mathbb{R}^3$ given by an equation $ax + by + cz + d = 0$, and assume without loos of generality that $c \neq 0$. Then

$$\boldsymbol{x}_x(x, y) = (1, 0, -a/c) \qquad \text{and} \qquad \boldsymbol{x}_y(x, y) = (0, 1, -b/c).$$

In particular, we have

$$E(x, y) = 1 + \frac{a^2}{c^2}, \qquad F(x, y) = \frac{ab}{c^2}, \qquad G(x, y) = 1 + \frac{b^2}{c^2}$$

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**Example 7.11. Coefficients of the first fundamental form for a graph of a function:** Let a surface be given by a graph of a function $g$, namely $\boldsymbol{x}(u,v) := (u,v,g(u,v)) = (u,v,u^2+v^2)$ for $(u,v) \in U := \mathbb{R}^2$. Then

$$\boldsymbol{x}_u(u,v) = (1,0,g_u) = (1,0,2u) \qquad \text{and} \qquad \boldsymbol{x}_v(u,v) = (0,1,g_v) = (0,1,2v).$$

In particular, we have

$$
\begin{aligned}
E &= (1,0,g_u) \cdot (1,0,g_u) = 1 + g_u^2, && \text{here} \quad E(u,v) = 1 + 4u^2, \\
F &= (1,0,g_u) \cdot (0,1,g_v) = g_u g_v, && \text{here} \quad F(u,v) = 8uv, \\
G &= (0,1,g_v) \cdot (0,1,g_v) = 1 + g_v^2 && \text{here} \quad G(u,v) = 1 + 4v^2,
\end{aligned}
$$

**Example 7.12. Coefficients of the first fundamental form for a surface of revolution:** Let $S$ be obtained by rotating the space curve given by $\boldsymbol{\alpha}(v) = (f(v),0,g(v))$, $v \in \mathbb{R}$, around the $z$-axis (without self-intersections and without meeting the $z$-axis, i.e., $f(v) = 0$). A parametrization is then given by

$$\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

$(u,v) \in (-\pi,\pi) \times \mathbb{R}$. Here, we have

$$\boldsymbol{x}_u(u,v) = (-f(v)\sin u, f(v)\cos u, 0) \quad \text{and} \quad \boldsymbol{x}_v(u,v) = (f'(v)\cos u, f'(v)\sin u, g'(v)).$$

The coefficients of the first fundamental form in this parametrization are

$$E(u,v) = f(v)^2, \quad F(u,v) = 0 \quad \text{and} \quad G(u,v) = |f'(v)|^2 + |g'(v)|^2 = \|\boldsymbol{\alpha}'(v)\|^2.$$

## 7.3 Arc lengths of a curve and angles between curves in a surface

The aim of the following remark is to calculate the arc length of a curve in a surface *using only the coefficients of the first fundamental form.*

**Definition 7.13.** Let $\boldsymbol{\alpha}\colon I \longrightarrow S$ be a curve on a regular surface $S$. Then the length of $\boldsymbol{\alpha}$, measured from a point $\boldsymbol{\alpha}(u_0)$ for some $u_0 \in I$, is

$$\ell(u) := \int_{u_0}^{u} \sqrt{\langle \boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle_{\boldsymbol{\alpha}(s)}}\, \mathrm{d}s.$$

**Proposition 7.14** (evident).

$$\ell(u) := \int_{u_0}^{u} [I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s))]^{1/2}\, \mathrm{d}s.$$

**Remark 7.15.** Let $\boldsymbol{\alpha}\colon I \longrightarrow S$ be a curve on a regular surface $S$ and $\boldsymbol{x}\colon U \longrightarrow S$ a local parametrization such that $\boldsymbol{\alpha}(I) \subset \boldsymbol{x}(U)$. Denote by $\boldsymbol{\beta} = (u,v)$ the corresponding curve in the parameter domain (i.e., $\boldsymbol{\alpha}(s) = \boldsymbol{x}(\boldsymbol{\beta}(s)) = \boldsymbol{x}(u(s),v(s))$).

Let $E, F, G$ be the coefficients of the first fundamental form w.r.t. the parametrization $\boldsymbol{x}$. Then the arc lengths of $\boldsymbol{\alpha}$ from $s_0 \in I$ to $s_1 \in I$ can be expressed in terms of $E, F, G$ only as follows:

$$\ell(s_1) = \int_{s_0}^{s_1} [I_{\boldsymbol{\alpha}(t)}(\boldsymbol{\alpha}'(t))]^{1/2}\, \mathrm{d}t = \int_{s_0}^{s_1} \sqrt{u'(t)^2 E(\boldsymbol{\beta}(t)) + 2u'(t)v'(t)F(\boldsymbol{\beta}(t)) + v'(t)^2 G(\boldsymbol{\beta}(t))}\, \mathrm{d}t.$$

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**Example 7.16. The hyperbolic plane.** We construct a surface by fixing the coefficients of the first fundamental form $E, F, G$ only. Actually, this is the first example which cannot (in total) be realized as a surface in $\mathbb{R}^3$.

Let $U := \{ (u, v) \in \mathbb{R}^2 \mid v > 0 \}$ be the upper halfplane and set

$$E(u, v) := \frac{1}{v^2}, \quad F(u, v) := 0 \quad \text{and} \quad G(u, v) := \frac{1}{v^2},$$

i.e., $F = 0$ and $E = G$.

Let us now assume that there is a surface $S$ in an ambient space $\mathbb{R}^n$ and a parametrization $\boldsymbol{x} \colon U \longrightarrow S$ such that the corresponding coefficients of the fundamental form have the desired form.

Consider a curve $\boldsymbol{\alpha} \colon (0, \infty) \longrightarrow S$ given by $\boldsymbol{\alpha}(s) = \boldsymbol{x}(0, s)$. In the coordinates on $U$, the curve has the form $\boldsymbol{\beta} \colon (0, \infty) \longrightarrow U$, $\boldsymbol{\beta}(s) = (0, s)$. Then

$$\|\boldsymbol{\alpha}'(s)\|^2 = 0E(0, s) + 0 + 1G(0, s) = \frac{1}{s^2}$$

Therefore, the arc length of $\boldsymbol{\alpha}$ from $\boldsymbol{\alpha}(a)$ to $\boldsymbol{\alpha}(b)$ on $S$ is

$$\int_a^b \|\boldsymbol{\alpha}'(s)\| \, \mathrm{d}s = \int_a^b \frac{1}{s} \, \mathrm{d}s = \log b - \log a = \log \frac{b}{a}.$$

The upper half-plane $U = \mathbb{R} \times (0, \infty)$ together with the first fundamental form above is called the *upper half-plane model of the hyperbolic plane*. The corresponding surface $S$, the *hyperbolic plane*, is sometimes denoted by $\mathbb{H}$.

**Example.** In the upper half-plane model of the hyperbolic plane, consider a curve $\alpha(\vartheta) = (r_0 \cos \vartheta, r_0 \sin \vartheta)$. Let us compute the length of thiscurve between $\vartheta_0$ and $\vartheta_1$:

$$l(\alpha) = \int_{\vartheta_0}^{\vartheta_1} \sqrt{I_{(r_0 \cos \vartheta, r_0 \sin \vartheta)}(-r_0 \sin \vartheta, r_0 \cos \vartheta)} d\vartheta = \int_{\vartheta_0}^{\vartheta_1} \frac{1}{r_0 \sin \vartheta} r_0 d\vartheta = \int_{\vartheta_0}^{\vartheta_1} \frac{1}{\sin \vartheta} d\vartheta = \log \tan \frac{\vartheta}{2} \Big|_{\vartheta_0}^{\vartheta_1},$$

which does not depend on $r_0$!

This is as expected, since $I_{(u,v)}(\alpha'(s)) = I_{(\lambda u, \lambda v)}([\lambda \alpha(s)]')$ in this case.

**Remark. Coordinate curves and angle.** Let $\boldsymbol{x} \colon U \longrightarrow S$ be a parametrization of a regular surface $S \subset \mathbb{R}^n$, $(u_0, v_0) \in U$. Consider the curves

$$\boldsymbol{\alpha}_1(s) = \boldsymbol{x}(u_0 + s, v_0) \qquad \text{and} \qquad \boldsymbol{\alpha}_2(s) = \boldsymbol{x}(u_0, v_0 + s)$$

with $s$ being small. These curves are called the *coordinate curves* of the parametrization $\boldsymbol{x}$. The angle formed by the two curves meeting in $(u_0, v_0)$ can be calculated by

$$\cos \vartheta = \frac{\boldsymbol{\alpha}_1'(0) \cdot \boldsymbol{\alpha}_2'(0)}{\|\boldsymbol{\alpha}_1'(0)\| \|\boldsymbol{\alpha}_2'(0)\|}.$$

But $\boldsymbol{\alpha}_1'(0) = \boldsymbol{x}_u(u_0, v_0)$ and $\boldsymbol{\alpha}_2'(0) = \boldsymbol{x}_v(u_0, v_0)$, so that (omitting the argument $(u_0, v_0)$)

$$\cos \vartheta = \frac{\boldsymbol{x}_u \cdot \boldsymbol{x}_v}{\|\boldsymbol{x}_u\| \|\boldsymbol{x}_v\|} = \frac{F}{\sqrt{EG}}.$$

## 7.4 Area of subsets of a surface

**Definition 7.17.** Let $R_0 \subset U$, $R = \boldsymbol{x}(R_0) \subset S$. The *area* of a region $R = \boldsymbol{x}(R_0)$ is defined as

$$\text{area}(R) := \int_{R_0} \sqrt{EG - F^2} \, du \, dv.$$

**Example 7.18.** Let $S$ be a half of a cylinder parametrized by

$$\boldsymbol{x}(u, v) = (u, v, \sqrt{1 - v^2}), \qquad (u, v) \in U = (-1, 1) \times (-1, 1)$$

Then $E \equiv 1$, $F \equiv 0$, $G = 1/(1 - v^2)$, so

$$\text{area}(S) = \int_U \sqrt{EG - F^2} \, du \, dv = \int_{-1}^1 du \int_{-1}^1 \sqrt{1/(1 - v^2)} \, dv = 2\pi$$

The definition of area depends at first sight on the local parametrization $\boldsymbol{x} \colon U \longrightarrow S$. Actually, it does not:

**Proposition 7.19.** Assume that we have two local parametrizations $\boldsymbol{x}_1 \colon U_1 \longrightarrow S$ and $\boldsymbol{x}_2 \colon U_2 \longrightarrow S$ with $\boldsymbol{x}_1(U_1) = \boldsymbol{x}_2(U_2) =: W$. Denote by $E_1, F_1, G_1$ and $E_2, F_2, G_2$ the coefficients of the first fundamental form in the parametrisation $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively.

Let $R \subset W$. Denote by $R_1 := \boldsymbol{x}_1^{-1}(R)$ and $R_2 := \boldsymbol{x}_2^{-1}(R)$ the corresponding regions in the respective parameter domains. Then

$$\int_{R_1} \sqrt{E_1 G_1 - F_1^2} \, du_1 \, dv_1 = \int_{R_2} \sqrt{E_2 G_2 - F_2^2} \, du_2 \, dv_2.$$

**Example 7.20.**

(a) **The sphere.** Let $S$ be the sphere of radius $r > 0$ in $\mathbb{R}^3$,

$$\boldsymbol{x}(u, v) = (r \cos u \sin v, r \sin u \sin v, r \cos v)$$

($v$ measures *latitude*, $u$ measures *longitude*, and $(u, v)$ are called *spherical coordinates*). We have

$$E(u, v) = r^2 \sin^2 v, \quad F(u, v) = 0 \quad \text{and} \quad G(u, v) = r^2,$$

so that $EG - F^2 = r^4 \sin^2 v$.

Let us compute the area of a "slice" of the sphere enclosed by planes $z = z_0$ and $z = z_1$, where $-r \le z_1 < z_0 \le r$. This corresponds to the domain $\arccos z_0 \le v \le \arccos z_1$, $u \in (0, 2\pi)$. Therefore the area is

$$\int_0^{2\pi} du \int_{\arccos z_0}^{\arccos z_1} r^2 \sin^2 v \, dv = 2\pi r^2 (z_0 - z_1).$$

(b) **Torus of revolution:** Consider the parametrization

$$\boldsymbol{x} \colon U := (0, 2\pi) \times (0, 2\pi) \longrightarrow S,$$
$$\boldsymbol{x}(u, v) := \big( (R + r \cos v) \cos u, (R + r \cos v) \sin u, r \sin v \big)$$

for $0 < r < R$. This surface is a surface of revolution, obtained by rotating the curve $\boldsymbol{\alpha}$ given by

$$\boldsymbol{\alpha}(v) = \big( (R + r \cos v), 0, r \sin v \big)$$

(which is a circle of radius $r$ in the $(x, z)$-plane centered at the point $(R, 0, 0)$) around the $z$-axis.
Then

$$\boldsymbol{x}_u(u, v) = \big(-(R + r\cos v)\sin u, (R + r\cos v)\cos u, 0\big),$$
$$\boldsymbol{x}_v(u, v) = \big(-r\sin v\cos u, -r\sin v\sin u, r\cos v\big)$$

and therefore

$$E(u, v) = (R + r\cos v)^2, \quad F(u, v) = 0 \quad \text{and} \quad G(u, v) = r^2.$$

In particular, $\sqrt{EG - F^2} = (R + r\cos v)r$, hence

$$\operatorname{area}(S) = \int_0^{2\pi} \int_0^{2\pi} (R + r\cos v)r \, du \, dv = 4\pi^2 rR.$$

(c) **Hyperbolic plane:** Recall that we have the parameter domain $U := \mathbb{R} \times (0, \infty)$ together with the coefficients of the fundamental form

$$E(u, v) = G(u, v) = \frac{1}{v^2}, \quad F(u, v) = 0,$$

and $\sqrt{EG - F}(u, v) = 1/v^2$. Let $R_{a,b} := (0, b) \times (a, 2a)$, then the corresponding region in the hyperbolic plane $\mathbb{H}$ has area

$$\operatorname{area}(R) = \int_{R_{a,b}} \frac{1}{v^2} \, du \, dv = \int_0^b du \int_a^{2a} \frac{1}{v^2} \, dv = b/2a.$$

In particular, if $b = a$, we obtain $1/2$ which does not depend on $a$.

# 8 Smooth maps between surfaces

Recall that $f \colon U \longrightarrow \mathbb{R}^m$ is smooth at $p \in U$ if all partial derivatives of $f$ at $p$ exist and are continuous. We need $U \subset \mathbb{R}^n$ to be *open* to be able to define a partial derivative.

Let $S \subset \mathbb{R}^n$ be a regular surface and $f \colon S \longrightarrow \mathbb{R}^m$. Since $S$ is not open in $\mathbb{R}^n$ ($n \geq 3$), we need to define smoothness of $f$ on $S$.

**Definition 8.1.** We say that $f \colon S \longrightarrow \mathbb{R}^m$ is smooth at $p$ if

$$f \circ \boldsymbol{x} \colon U \longrightarrow \mathbb{R}^m$$

is smooth at $q$ where $\boldsymbol{x} \colon U \longrightarrow S$ is a parametrization with $\boldsymbol{x}(q) = p$.

**Remark 8.2.** This definition does not depend on the parametrization $\boldsymbol{x}$. Indeed, if $\boldsymbol{y} \colon V \longrightarrow S$ is another parametrization (assume that $\boldsymbol{x}(U) = \boldsymbol{y}(V)$), then there exists a diffeomorphism $h \colon U \longrightarrow V$ such that $\boldsymbol{y} = \boldsymbol{x} \circ h$ (change of parameter). In particular, $f \circ \boldsymbol{y} = (f \circ \boldsymbol{x}) \circ h$ is also smooth.

## 8.1 The Gauss map

Let $S$ be a regular surface in $\mathbb{R}^3$.

**Definition 8.3.** The *Gauss map*

$$\boldsymbol{N} \colon S \longrightarrow S^2$$

assigns, to each point $p \in S$, the unit normal to $S$ at $p$, i.e., the unit vector orthogonal to $T_pS \subset \mathbb{R}^3$ (which is determined up to sign only!). Here, $S^2 := \{\, (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1 \,\}$ is the unit sphere in $\mathbb{R}^3$.

In a local parametrization $\boldsymbol{x} \colon U \longrightarrow S$ of $S$, we have

$$\boldsymbol{N} \circ \boldsymbol{x}(u, v) := \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}(u, v),$$

and this map is always smooth.

**Example 8.4.**

(a) **Plane in $\mathbb{R}^3$:** $S = \{\, (x, y, z)\mathbb{R}^3 \,|\, ax + by + cz + d = 0 \,\}$. Then $\boldsymbol{N} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}} \equiv \mathrm{const}.$

(b) **Graph of a function:** $S = \{\, (u, v, g(u, v)) \,|\, (u, v) \in U \,\}$, $g \colon U \longrightarrow \mathbb{R}$ smooth, then $\boldsymbol{x}_u = (1, 0, g_u)$, $\boldsymbol{x}_v = (0, 1, g_v)$, then the Gauss map is given by $\boldsymbol{N} \colon S \longrightarrow S^2$

$$\boldsymbol{N} \circ \boldsymbol{x} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|} = \frac{1}{\sqrt{1 + (g_u)^2 + (g_v)^2}}(-g_u, -g_v, 1).$$

As an example, take $g(u, v) = u^2 + v^2$, then

$$\boldsymbol{N}(\boldsymbol{x}(u, v)) = \frac{1}{\sqrt{1 + 4u^2 + 4v^2}}(-2u, -2v, 1)$$

Also, $S = f^{-1}(0)$ for $f(x, y, z) = x^2 + y^2 - z$, so $\nabla f = (2x, 2y, -1)$ is proportional to $\boldsymbol{N}$ as expected.

(c) **The catenoid:** $\boldsymbol{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$, then

$$\boldsymbol{x}_u(u, v) = (-\cosh v \sin u, \cosh v \cos u, 0) \quad \text{and} \quad \boldsymbol{x}_v(u, v) = (\sinh v \cos u, \sinh v \sin u, 1)$$

so that

$$(\boldsymbol{x}_u \times \boldsymbol{x}_v)(u, v) = (\cosh v \cos u, \cosh v \sin u, -\cosh v \sinh v),$$

and therefore

$$\boldsymbol{N}(\boldsymbol{x}(u, v)) = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v).$$

(d) **The sphere:** $\boldsymbol{N} \colon S^2 \longrightarrow S^2$ is given by $\boldsymbol{N}(p) = p$.

**Remark.** The Gauss map is well defined on $\boldsymbol{x}(U)$, but we may not be able to define it (continuously) on all $S$

**Example 8.5. Möbius band**

**Definition 8.6.** A surface in $\mathbb{R}^3$ is *non-orientable* if it is not possible to define the Gauss map globally.

**Example 8.7. Further maps on surfaces.** Let $S \subset \mathbb{R}^3$ be a surface.

(a) **Height function.** Fix $v \in S^2$, and define a function $h \colon S \longrightarrow \mathbb{R}$ by $h(p) := p \cdot v$. Then $h$ is smooth. You can think of $h$ measuring the height of $S$ if you stand on the plane orthogonal to $v$ fixed e.g. at the origin of $\mathbb{R}^3$.

(b) **Distance squared function.** Let $a \in \mathbb{R}^3$ and define $d^2 \colon S \longrightarrow \mathbb{R}$ by $d^2(p) := \|p - a\|^2 = (p - a) \cdot (p - a)$, then $d^2$ is smooth. ($d$ measures the distance of $p$ from $a$ *in the ambient space* $\mathbb{R}^3$).

## 8.2   The derivative of a smooth map between surfaces

**Definition 8.8.** Let $S$ be a regular surface in $\mathbb{R}^\ell$, $p \in S$ and $f \colon S \longrightarrow \mathbb{R}^m$ a smooth map. The *derivative of $f$ at $p$* is a linear map

$$d_p f \colon T_p S \longrightarrow \mathbb{R}^m$$

such that

$$d_p f(\boldsymbol{x}_u) = \partial_u (f \circ \boldsymbol{x})(q) \quad \text{and} \quad d_p f(\boldsymbol{x}_v) = \partial_v (f \circ \boldsymbol{x})(q)$$

for a local parametrization $\boldsymbol{x} \colon U \longrightarrow S$ of $S$ with $\boldsymbol{x}(q) = p$, $q \in U \subset \mathbb{R}^2$. For short, we write

$$\boldsymbol{f}_u := d_p f(\boldsymbol{x}_u) \quad \text{and} \quad \boldsymbol{f}_v := d_p f(\boldsymbol{x}_v),$$

suppressing the local parametrisation $\boldsymbol{x}$ in the notation $\boldsymbol{f}_u$ and $\boldsymbol{f}_v$.

**Remark 8.9.**

(a) As $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ is a basis of $T_p S$, and $\boldsymbol{w} \in T_p$ can be written as $\boldsymbol{w} = a\boldsymbol{x}_u + \boldsymbol{x}_v$, we have

$$d_p f(\boldsymbol{w}) = d_p f(a\boldsymbol{x}_u + b\boldsymbol{x}_v) = a d_p f(\boldsymbol{x}_u) + b d_p f(\boldsymbol{x}_v)$$

by the linearity of $d_p f$.

(b) $d_p f$ does not depend on the choice of local parametrization $\boldsymbol{x}$. Indeed, if we take $\boldsymbol{w} \in T_p S$ and compute its image, then if $\boldsymbol{w} = \boldsymbol{\alpha}'(0)$ for $\boldsymbol{\alpha} \colon I \longrightarrow S$ a smooth curve, $\boldsymbol{\alpha}(0) = p$, we have $d_p f(\boldsymbol{w}) = (f \circ \alpha)'(0))$.

**Example 8.10.**

(a) Let $S = \{(u, v, f(u,v)) \in \mathbb{R}^3\}$, where $f(u,v) = (u, v, u^2 + v^2)$ be a paraboloid with $\boldsymbol{x}(q) = \boldsymbol{p} = (0,1)$. Let $\boldsymbol{w} = (1,1)$. Then we can compute $d_p f(\boldsymbol{w})$ as follows.

The vector $\boldsymbol{w}$ is the tangent to the curve $\boldsymbol{\alpha} \colon (-\epsilon, \epsilon) \to S = \mathbb{R}^2$ $\boldsymbol{\alpha}(s) = (s, s + 1)$ at the point $\boldsymbol{\alpha}(0) = \boldsymbol{p} = (0,1)$ as $\boldsymbol{\alpha}' = \boldsymbol{w} = (1,1)$.

$f \circ \boldsymbol{\alpha}(s) = f(s, s+1) = (s, s+1, s^2 + (s+1)^2) = (s, s+1, 2s^2 + 2s + 1)$.
Hence, $(f \circ \boldsymbol{\alpha})'\big|_{s=0} = (1, 1, 4s+2)\big|_{s=0} = (1,1,2)$.

On the other hand,

$$\frac{\partial}{\partial u}(f \circ \boldsymbol{x})(q) = \frac{\partial}{\partial u}(u, v, u^2 + v^2)\,|_{u=0, v=1} = (1, 0, 2u)\,|_{u=0, v=1} = (1, 0, 0),$$

$$\frac{\partial}{\partial v}(f \circ \boldsymbol{x})(q) = \frac{\partial}{\partial v}(u, v, u^2 + v^2)\,|_{u=0, v=1} = (0, 1, 2v)\,|_{u=0, v=1} = (0, 1, 2).$$

So, $d_p f(\boldsymbol{w}) = 1 \cdot d_p f(\boldsymbol{x}_u) + 1 \cdot d_p f(\boldsymbol{x}_v) = (1, 0, 0) + 0, 1, 2) = (1, 1, 2)$.

(b) Let $S = \{\, (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \,\}$ be a cylinder in $\mathbb{R}^3$ and $f \colon S \longrightarrow \mathbb{R}$ be given by $f(p) = p \cdot p = \|p\|^2$. A local parametrization of $S$ is given by

$$\boldsymbol{x} \colon U \longrightarrow S, \qquad \boldsymbol{x}(\vartheta, z) = (\cos \vartheta, \sin \vartheta, z), \qquad (\vartheta, r) \in U$$

Here, at least two parameter domains $U_1 = (0, 2\pi) \times \mathbb{R}$ and $U_2 = (-\pi, \pi) \times \mathbb{R}$ are needed in order to cover the entire cylinder. Then we have $(f \circ \boldsymbol{x})(\vartheta, z) = f(\cos \vartheta, \sin \vartheta, z)$ and

$$d_p f(\boldsymbol{x}_\vartheta) = \boldsymbol{f}_\vartheta = \frac{\partial}{\partial \vartheta}(f \circ \boldsymbol{x}) = 0 \quad \text{and} \quad d_p f(\boldsymbol{x}_z) = \boldsymbol{f}_z = \frac{\partial}{\partial z}(f \circ \boldsymbol{x}) = 2z.$$

(c) **(Gauss map of a catenoid)** Let $S$ be parametrized by

$$\boldsymbol{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v),$$

then its Gauss map is given by

$$\boldsymbol{N}(\boldsymbol{x}(u, v)) = \frac{1}{\cosh v}\big(\cos u, \sin u, -\sinh v\big).$$

In particular, the derivative is

$$d_p\boldsymbol{N}(\boldsymbol{x}_u) = \boldsymbol{N}_u = \frac{1}{\cosh v}\big(-\sin u, \cos u, 0\big) \quad \text{and}$$

$$d_p\boldsymbol{N}(\boldsymbol{x}_v) = \boldsymbol{N}_v = \frac{1}{\cosh^2 v}\big(-\cos u \sinh v, -\sin u \sinh v, -1\big).$$

**Proposition 8.11 (Chain Rule).** Let $f\colon S_1 \longrightarrow S_2$ and $g\colon S_2 \longrightarrow S_3$ be smooth maps between the surfaces $S_1$, $S_2$ and $S_3$, then $g \circ f\colon S_1 \longrightarrow S_3$ is smooth and its derivative is given by

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f\colon T_p S_1 \longrightarrow T_{g(f(p))}S_3$$

as linear maps, or pointwise,

$$d_p(g \circ f)(\boldsymbol{w}) = d_{f(p)}g(d_p f(\boldsymbol{w}))$$

for all $\boldsymbol{w} \in T_p S_1$ and $p \in S_1$.

**Example.** Let $S_1 = \mathbb{R}^2$, $S_2 = \mathbb{R}^2$ be two copies of a plane and $S_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2, z \geq 0\}$ be a cone. Let $(r, \varphi)$ be polar coordinates on $S_1$. Consider the maps $S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3$, where $f(r, \varphi) = (2r, \varphi)$ and $g(r, \varphi) = (r \cos \varphi, r \sin \varphi, R)$. Let us verify that $d_p(g \circ f) = d_{f(p)}g \circ d_p f$.

Indeed,

$$\begin{pmatrix} 2\cos\varphi & -2r\sin\varphi \\ 2\sin\varphi & 2r\cos\varphi \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -2r\sin\varphi \\ \sin\varphi & 2r\cos\varphi \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

## 8.3 Isometries and conformal maps

Let $S \subset \mathbb{R}^\ell$ be a regular surface. Recall that the *first fundamental form* (1$^{\text{st}}$FF) is given by

$$I_p\colon T_p S \longrightarrow \mathbb{R}, \qquad I_p(\boldsymbol{w}) = \langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\mathbb{R}^\ell} = \|\boldsymbol{w}\|_{\mathbb{R}^\ell}^2.$$

Recall also that the 1$^{\text{st}}$FF is needed to calculate

- lengths of curves in $S$,

- angles between curves in $S$ and

- the area of subsets of $S$.

Let now $S$ and $\widetilde{S}$ be two surfaces with 1$^{\text{st}}$FFs $I$ and $\widetilde{I}$, respectively, let $f\colon S \longrightarrow \widetilde{S}$ be a smooth map. If $d_p f\colon T_p S \longrightarrow T_{f(p)}\widetilde{S}$ "preserves" $I_p$ and $\widetilde{I}_{f(p)}$, then these calculations should give the same result, i.e., $S$ and $\widetilde{S}$ are basically the same from a metric point of view (at least locally: see Example 8.13 (a) below)

**Definition 8.12.** Let $f\colon S \longrightarrow \widetilde{S}$ be a smooth map between two surfaces $S$ and $\widetilde{S}$.

(a) The map $f$ is called a *(local) isometry* if

$$\langle d_p f(\boldsymbol{w}_1), d_p f(\boldsymbol{w}_2)\rangle_{f(p)} = \langle \boldsymbol{w}_1, \boldsymbol{w}_2\rangle_p$$

for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_p S$ and $p \in S$. The surfaces $S$ and $\widetilde{S}$ are called *(locally) isometric* if there is a (local) isometry between them.

(b) The map $f$ is called a *(global) isometry* if $f$ is a local isometry and, additionally, $f \colon S \longrightarrow \widetilde{S}$ is *bijective*.

The surfaces $S$ and $\widetilde{S}$ are called *(globally) isometric* if there is a (global) isometry between them.

(c) The map $f$ is called *conformal* if there is a smooth function

$$\lambda \colon S \longrightarrow (0, \infty)$$

such that

$$\langle d_p f(\boldsymbol{w}_1), d_p f(\boldsymbol{w}_2)\rangle_{f(p)} = \lambda(p)\langle \boldsymbol{w}_1, \boldsymbol{w}_2\rangle_p$$

for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_p S$ and $p \in S$.

The surfaces $S$ and $\widetilde{S}$ are called *conformally equivalent* if there is a conformal map between them.

**Remark.**

(a) Given a symmetric bilinear form $\langle, \rangle$, one can write $\langle \boldsymbol{w}_1, \boldsymbol{w}_2\rangle = \frac{1}{2}(\|\boldsymbol{w}_1 + \boldsymbol{w}_2\|^2 - \|\boldsymbol{w}_1\|^2 - \|\boldsymbol{w}_2\|^2)$, which means that being a local isometry is equivalent to preserving 1$^\text{st}$FF, i.e. $\widetilde{I}_{f(p)}(d_p f(\boldsymbol{w})) = I_p(\boldsymbol{w})$, cf. Prop. 8.15.

(b) A conformal map with $\lambda \equiv 1$ is obviously a local isometry.

(c) A global isometry is obviously a local isometry, but not vice versa (see Example 8.13 (c) below).

(d) Conformal maps preserve angles. Indeed,

$$\vartheta = \angle(\boldsymbol{w}_1, \boldsymbol{w}_2) := \frac{\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle_p}{\|\boldsymbol{w}_1\|_p \|\boldsymbol{w}_1\|_p} \quad \text{and}$$

$$\angle(d_p f(\boldsymbol{w}_1), d_p f(\boldsymbol{w}_2)) := \frac{\langle d_p f(\boldsymbol{w}_1), d_p f(\boldsymbol{w}_1)\rangle_p}{\|d_p f(\boldsymbol{w}_1)\|_p \|d_p f(\boldsymbol{w}_1\|_p)} = \frac{\lambda(p)\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle_p}{\sqrt{\lambda(p)}\|\boldsymbol{w}_1\|_p \sqrt{\lambda(p)}\|\boldsymbol{w}_1\|_p} = \vartheta$$

since the factors involving $\lambda(p) > 0$ cancel each other.

(e) Local isometries preserve lengths of curves (but not distances between points). Global isometries preserve distances.

**Example 8.13.**

(a) Let $S = (0, 2\pi) \times \mathbb{R}$ and $\widetilde{S} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ (a cylinder). Define $f \colon S \longrightarrow \widetilde{S}$ by $f(\vartheta, z) = (\cos \vartheta, \sin \vartheta, z)$ for $p = (\vartheta, z) \in S$. We can think of $S$ as being parametrized by itself (as a subset of the plane $\mathbb{R}^2$), and $T_p S = \mathbb{R}^2$.

One way to show that $f$ is a local isometry is to ensure that it preserves 1$^\text{st}$FF (the identity matrix), which is an elementary computation of $\boldsymbol{f}_\vartheta$ and $\boldsymbol{f}_z$ and their dot products, cf. Prop. 8.15.

Alternatively, one can compute the differential of $f$ explicitly. Write $\boldsymbol{w} = (a, b) \in T_p S$. We need $\boldsymbol{\alpha} \colon I \longrightarrow S$ with $I$ being an open interval containing 0, $\boldsymbol{\alpha}(0) = p$ and $\boldsymbol{\alpha}'(0) = \boldsymbol{w}$. Take a line through $p \in S \subset \mathbb{R}^2$ in direction $\boldsymbol{w}$, i.e.

$$\boldsymbol{\alpha}(t) = p + t\boldsymbol{w} = (\vartheta + ta, z + tb).$$

Then

$$d_p f(\boldsymbol{w}) = d_p f(\boldsymbol{\alpha}'(0)) = (f \circ \boldsymbol{\alpha})'(0)$$

Here, we have

$$(f \circ \boldsymbol{\alpha})(t) = \big(\cos(\vartheta ta), \sin(\vartheta + ta), z + tb\big),$$

so that

$$(f \circ \boldsymbol{\alpha})'(0) = (-a \sin \vartheta, a \cos \vartheta, b) = d_p f(\boldsymbol{w}).$$

Now,

$$\langle d_p f(\boldsymbol{w}), d_p f(\boldsymbol{w}) \rangle_{f(p)} = \langle (-a \sin \vartheta, a \cos \vartheta, b), (-a \sin \vartheta, a \cos \vartheta, b) \rangle = a^2 + b^2,$$

but we also have $\langle \boldsymbol{w}, \boldsymbol{w} \rangle_p = a^2 + b^2$, hence $f$ is a local isometry.

(b) If we consider $f \colon S \longrightarrow \{\, (x, y, z) \mid x^2 + y^2 = 1, (x, y) \neq (1, 0) \,\}$, then $f$ is bijective (check this!) and $f$ is indeed a *global* isometry.

(c) If we consider $f \colon \mathbb{R} \times \mathbb{R} \longrightarrow \widetilde{S}$ (with the same definition of $f(\vartheta, z)$ as before, but now $\vartheta \in \mathbb{R}$), then $f$ is still a local isometry (the calculation remains the same as above), but not a *global* isometry: $f$ is no longer injective and hence not bijective.

**Example 8.14 (Conformal bijections of $\mathbb{R}^2$).** As one can recall from Complex Analysis, conformal maps are holomorphic (or anti-holomorphic) and vise versa. Thus conformal bijections of the plane are holomorphic one-to-one maps. They must have a single pole at infinity, so they are polynomial of degree one (possibly with conjugation), i.e. $f(z) = az + b$ or $f(z) = a\bar{z} + b$, $a, b \in \mathbb{C}$, $a \neq 0$. The conformal factor is $\lambda(z) = |a|^2$.

**Proposition 8.15.** Let $S$, $\widetilde{S}$ be two surfaces and $\boldsymbol{x} \colon U \longrightarrow S$ be a local parametrization of $S$.

A map $f \colon S \longrightarrow \widetilde{S}$ is a local isometry on $\boldsymbol{x}(U)$ if and only if

$$\langle \boldsymbol{f}_u, \boldsymbol{f}_u \rangle = E, \quad \langle \boldsymbol{f}_u, \boldsymbol{f}_v \rangle = F \quad \text{and} \quad \langle \boldsymbol{f}_v, \boldsymbol{f}_v \rangle = G, \tag{8.-2}$$

where $E, F, G$ are the coefficients of the 1$^{\text{st}}$FF w.r.t. $\boldsymbol{x}$. Here $\boldsymbol{f}_u = \partial_u(f \circ \boldsymbol{x})$ and $\boldsymbol{f}_v = \partial_v(f \circ \boldsymbol{x})$ and $(u, v) \in U$ are the parameter coordinates).

**Remark.**

(a) If we denote by $\widetilde{E}$, $\widetilde{F}$ and $\widetilde{G}$ the coefficients of the 1$^{\text{st}}$FF of $\widetilde{S}$ w.r.t. the parametrization $\widetilde{\boldsymbol{x}} = f \circ \boldsymbol{x} \colon U \longrightarrow \widetilde{S}$, then we can rephrase this as

$$\widetilde{E} = E, \quad \widetilde{F} = F \quad \text{and} \quad \widetilde{G} = G.$$

(b) A similar result holds for conformal maps: $f$ is conformal on $\boldsymbol{x}(U)$ iff there exists a smooth map $\lambda \colon U \longrightarrow (0, \infty)$ such that

$$\langle \boldsymbol{f}_u, \boldsymbol{f}_u \rangle = \lambda E, \quad \langle \boldsymbol{f}_u, \boldsymbol{f}_v \rangle = \lambda F \quad \text{and} \quad \langle \boldsymbol{f}_v, \boldsymbol{f}_v \rangle = \lambda G,$$

**Example 8.16.** (a) Spheres of distinct radii are conformally equivalent (but not isometric, will see this later).

(b) **Gauss map of the catenoid is conformal.** We have seen in Example 8.10 (c) and previous examples that for the parametrization $\boldsymbol{x}$ given by

$$\boldsymbol{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v),$$

the coefficients of the $1^{\text{st}}$FF are

$$E = G = \cosh^2 v \quad \text{and} \quad F = 0.$$

Moreover, the derivatives of the Gauss map are

$$\boldsymbol{N}_u = \frac{1}{\cosh v} \begin{pmatrix} -\sin u \\ \cos u \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{N}_v = \frac{1}{\cosh^2 v} \begin{pmatrix} -\cos u \sinh v \\ -\sin u \sinh v \\ -1 \end{pmatrix}.$$

Now,

$$\langle \boldsymbol{N}_u, \boldsymbol{N}_u \rangle = \frac{1}{\cosh^2 v} = \frac{1}{\cosh^4 v} E, \quad \langle \boldsymbol{N}_u, \boldsymbol{N}_v \rangle = 0 = F \quad \text{and}$$

$$\langle \boldsymbol{N}_v, \boldsymbol{N}_v \rangle = \frac{\sinh^2 v + 1}{\cosh^4 v} = \frac{1}{\cosh^2 v} = \frac{1}{\cosh^4 v} G,$$

so $\boldsymbol{N}$ is a conformal map with conformal factor (in local parametrization) $\mu$ given by $\mu(u, v) = 1/\cosh^4(v)$.

**Remark 8.17.** A map $f : S \to \widetilde{S}$ that preserves the lengths of all curves is a local isometry. Example c (f: plane $\to$ cylinder) shows that it is not necessirily glogal isometry.

# 9 Geometry of the Gauss map

## 9.1 The Weingarten map

**Lemma 9.1.** Let $S$ be a surface in $\mathbb{R}^3$ and $\boldsymbol{N} : S \longrightarrow S^2$ be its Gauss map. Then $d_p\boldsymbol{N}(\boldsymbol{w})$ is orthogonal to $\boldsymbol{N}(p)$ for every $\boldsymbol{w} \in T_pS$. In particular, we can identify $T_{\boldsymbol{N}(p)}S^2$ and $T_pS$, and consider $d_p\boldsymbol{N}$ as a map

$$d_p\boldsymbol{N} : T_pS \longrightarrow T_pS.$$

Moreover, $d_p\boldsymbol{N}$ is *symmetric*, i.e.,

$$\langle d_p\boldsymbol{N}(\boldsymbol{w}_1), \boldsymbol{w}_2 \rangle = \langle \boldsymbol{w}_1, d_p\boldsymbol{N}(\boldsymbol{w}_2) \rangle$$

for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_pS$.

**Definition 9.2.** (a) The map $-d_p\boldsymbol{N} : T_pS \longrightarrow T_pS$ is called the *Weingarten map* of the surface $S \subset \mathbb{R}^3$ at $p \in S$.

(b) The quadratic form $II_p : T_pS \longrightarrow \mathbb{R}$, $II_p(\boldsymbol{w}) = \langle -d_p\boldsymbol{N}(\boldsymbol{w}), \boldsymbol{w} \rangle$, is called the *second fundamental form of S at p*.

**Remark 9.3.** Since $-d_p\boldsymbol{N}$ is symmetric, the Weingarten map is diagonalizable in an orthogonal basis of $T_pS$.

Since $-d_p\boldsymbol{N}$ is now a linear operator on the tangent space $T_pS$, we can calculate its characteristic polynomial, trace, determinant and eigenvalues (these do not depend on a basis).

**Definition 9.4.** Let $S$ be a regular surface in $\mathbb{R}^3$ with Gauss map $\boldsymbol{N} \colon S \longrightarrow S^2$ and Weingarten map $-d_p\boldsymbol{N} \colon T_pS \longrightarrow T_pS$ at $p \in S$.

(a) $K(p) = \det(-d_p\boldsymbol{N})$ is called the *Gauss curvature of $S$ at $p$*.

(b) $H(p) = \frac{1}{2}\operatorname{tr}(-d_p\boldsymbol{N})$ is called the *mean curvature of $S$ at $p$*.

(c) The eigenvalues $\kappa_1(p)$, $\kappa_2(p)$ of $-d_p\boldsymbol{N}$ are called *principal curvatures of $S$ at $p$*.

(d) The eigenvectors $\boldsymbol{e}_1(p)$, $\boldsymbol{e}_2(p)$ of $-d_p\boldsymbol{N}$ are called *principal directions of $S$ at $p$* (i.e., $-d_p\boldsymbol{N}(\boldsymbol{e}_i(p)) = \kappa_i(p)\boldsymbol{e}_i(p)$).

**Remark 9.5.** Obviously, we have

$$K(p) = \kappa_1(p)\kappa_2(p), \quad H(p) = \frac{1}{2}\big(\kappa_1(p) + \kappa_2(p)\big).$$

**Example 9.6** (Sphere). Let $S = S^2(r)$ for some $r > 0$ be a sphere. The normal vector at $\boldsymbol{p} \in S$ is given by

$$\boldsymbol{N}(\boldsymbol{p}) = \frac{1}{r}\,\boldsymbol{p}.$$

Thus, the Weingarten map is a scalar operator

$$-d_{\boldsymbol{p}}\boldsymbol{N}(\boldsymbol{w}) = -\frac{1}{r}\,\boldsymbol{w}.$$

In particular, the second fundamental form is

$$II_p(\boldsymbol{w}) = \langle -d_p\boldsymbol{N}(\boldsymbol{w}), \boldsymbol{w}\rangle = -\frac{1}{r}\|\boldsymbol{w}\|^2.$$

Moreover, the eigenvalues are $\kappa_1(p) = \kappa_2(p) = -1/r$, the Gauss curvature is $K(p) = 1/r^2$ and the mean curvature is $H(p) = -1/r$.

**Definition 9.7.** Let $S$ be a regular surface in $\mathbb{R}^3$ with Gauss map $\boldsymbol{N} \colon S \longrightarrow S^2$, and let $\boldsymbol{x} \colon U \longrightarrow S$ be a local parametrization. We call

$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N}, \qquad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} \quad \text{and} \quad N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N}$$

the *coefficients of the second fundamental form*.

**Proposition 9.8.** $L, M, N$ are indeed the coefficients of $II_p$ in the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$, i.e.

$$II_p(a\boldsymbol{x}_u + b\boldsymbol{x}_v) = a^2 L + 2abM + b^2 N$$

Computing the matrix of the Weingarten map in the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ gives a matrix

$$-d_p\boldsymbol{N} = \frac{1}{EG - F^2}\begin{pmatrix} GL - FM & GM - FN \\ -FL + EM & -FM + EN \end{pmatrix},$$

which results in the following.

**Proposition 9.9.**

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2}.$$

**Example 9.10. Hyperbolic paraboloid.**
Let $S := \{(x, y, z) \mid x^2 - y^2 + z = 0\}$. It may be parametrized as a graph of a function $z = f(x, y) = y^2 - x^2$, i.e., $\boldsymbol{x}(u, v) = (u, v, v^2 - u^2)$ for $(u, v) \in U = \mathbb{R}^2$. Then

$$\boldsymbol{x}_u = (1, 0, -2u), \qquad \boldsymbol{x}_v = (0, 1, 2v),$$
$$\boldsymbol{x}_{uu} = (0, 0, -2), \quad \boldsymbol{x}_{uv} = (0, 0, 0), \quad \boldsymbol{x}_{vv} = (0, 0, 2).$$

We also need the normal and calculate

$$\boldsymbol{x}_u \times \boldsymbol{x}_v = (2u, -2v, 1),$$

which has norm $D = (4u^2 + 4v^2 + 1)^{1/2}$, hence

$$\boldsymbol{N} \circ \boldsymbol{x} = \frac{1}{D}(2u, -2v, 1).$$

The coefficients of the 1$^{\text{st}}$FF and 2$^{\text{nd}}$FF are

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u = 1 + 4u^2, \quad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = -4uv, \quad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v = 1 + 4v^2$$
$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = \frac{-2}{D}, \quad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = 0, \quad N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = \frac{2}{D}.$$

Now,

$$EG - F^2 = (1 + 4u^2)(1 + 4v^2) - 16u^2v^2 = 1 + 4u^2 + 4v^2 = D^2 \quad \text{and} \quad LN - M^2 = \frac{-4}{D^2},$$

so that the Gauss curvature is
$$K = \frac{LN - M^2}{EG - F^2} = \frac{-4}{D^4} < 0$$

and the mean curvature is

$$H = \frac{EN + GL}{2(EG - F^2)} = \frac{(1 + 4u^2) - (1 + 4v^2)}{D^3} = \frac{4(u^2 - v^2)}{D^3}.$$

Let us calculate the principal curvatures at $\boldsymbol{x}(0, 0) = (0, 0, 0)$ (i.e., $(u, v) = (0, 0)$). Here, $K = -4$ and $H = 0$, hence we look for the roots $\kappa$ of

$$\kappa^2 - 2H\kappa + K = 0, \quad \text{or,} \quad \kappa^2 - 4 = 0,$$

i.e., $\kappa_1 = 2$ and $\kappa_2 = -2$.

**Definition 9.11.** A parametrization $\boldsymbol{x}$ with $F = 0$ is called *orthogonal*, a parametrization $\boldsymbol{x}$ with $F = 0$ and $M = 0$ is called *principal*.

**Proposition 9.12.** Assume that the parametrization $\boldsymbol{x}$ of a surface is principal (i.e., $F = 0$ and $M = 0$), then $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are the principal directions. Moreover, the principal curvatures are

$$\kappa_1 = \frac{L}{E} \quad \text{and} \quad \kappa_2 = \frac{N}{G}.$$

Hence, the Gauss and mean curvatures are

$$K = \kappa_1\kappa_2 = \frac{LN}{EG} \quad \text{and} \quad H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{GL + EN}{2EG}.$$

**Example 9.13. Surface of revolution.** Let $S$ be obtained by rotating the curve given by $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$, $v \in I$ (some open interval) around the $z$-axis. Let us assume that $f(v) > 0$. A local parametrization is then given by

$$\boldsymbol{x}(u, v) = \begin{pmatrix} f(v)\cos u \\ f(v)\sin u \\ g(v) \end{pmatrix}$$

for $(u, v) \in U_1 = (0, 2\pi) \times I$ (and $(u, v) \in U_2 = (-\pi, \pi) \times I$ to cover the surface entirely). The derivatives are

$$\boldsymbol{x}_u = \begin{pmatrix} -f(v)\sin u \\ f(v)\cos u \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}_v = \begin{pmatrix} f'(v)\cos u \\ f'(v)\sin u \\ g'(v) \end{pmatrix}.$$

For the coefficients of the second fundamental form, we also need the *second derivatives* of $\boldsymbol{x}$:

$$\boldsymbol{x}_{uu} = \begin{pmatrix} -f(v)\cos u \\ -f(v)\sin u \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_{uv} = \boldsymbol{x}_{vu} = \begin{pmatrix} -f'(v)\sin u \\ f'(v)\cos u \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}_{vv} = \begin{pmatrix} f''(v)\cos u \\ f''(v)\sin u \\ g''(v) \end{pmatrix}.$$

The normal vector at $p = \boldsymbol{x}(u, v)$ is

$$\boldsymbol{N}(p) = \Big( \frac{1}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|} \boldsymbol{x}_u \times \boldsymbol{x}_v \Big)(u, v) = \frac{1}{\alpha'(v)} \begin{pmatrix} g'(v)\cos u \\ g'(v)\sin u \\ -f'(v) \end{pmatrix},$$

where $\|\boldsymbol{\alpha}'(v)\| = (f'(v)^2 + g'(v)^2)^{1/2}$. Now, the coefficients of the second fundamental form are

$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = \frac{-fg'}{\|\boldsymbol{\alpha}'\|}, \qquad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = 0 \quad \text{and}$$

$$N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = \frac{f''g' - f'g''}{\|\boldsymbol{\alpha}'\|}.$$

The coefficients of the 1$^{\text{st}}$FF

$$E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle = f^2, \quad F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = 0 \quad \text{and} \quad G = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = \|\boldsymbol{\alpha}'\|^2.$$

Now we can calculate all the curvatures. The principal curvatures are

$$\kappa_1 = \frac{L}{E} = \frac{-fg'}{f^2\|\boldsymbol{\alpha}'\|} = \frac{-g'}{f\|\boldsymbol{\alpha}'\|} \quad \text{and} \quad \kappa_2 = \frac{N}{G} = \frac{f''g' - f'g''}{\|\boldsymbol{\alpha}'\|^3}.$$

Hence, the Gauss and mean curvatures are

$$K = \kappa_1\kappa_2 = \frac{LN}{EG} = \frac{-g'(f''g' - f'g'')}{f\|\boldsymbol{\alpha}'\|^4} \quad \text{and}$$

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{-g'}{2f} + \frac{f''g' - f'g''}{2\|\boldsymbol{\alpha}'\|^3}.$$

**Example 9.14. Torus of revolution.** Apply the above to the case $f(v) = R + r\cos(v/r)$ and $g(v) = r\sin(v/r)$, $0 < r < R$. Calculate the principal, Gauss curvature and mean curvatures.

We just calculate

$$f'(v) = -\sin(v/r), \qquad\qquad g'(v) = \cos(v/r),$$
$$f''(v) = -\frac{1}{r}\cos(v/r), \qquad\qquad g''(v) = -\frac{1}{r}\sin(v/r).$$

so that

$$\kappa_1 = \frac{-g'}{f} = \frac{\cos}{R + r\cos} \quad\text{and}\quad \kappa_2 = \frac{f''g' - f'g''}{f} = -\frac{1}{r}(\cos^2 + \sin^2) = -\frac{1}{r}$$

since $(f')^2 + (g')^2 = 1$ (the arguments of cos and sin in this formula are $v/r$). In particular, one principal curvature is constant (it is the one coming from going around the torus along the small circle, i.e., in direction $\boldsymbol{x}_u$). Moreover,

$$K = \kappa_1\kappa_2 = \frac{\cos}{r(R + r\cos)} \quad\text{and}\quad H = \frac{\cos}{2(R + r\cos)} - \frac{1}{2r} = \frac{-R}{2r(R + r\cos)}.$$

Note that the mean curvature never vanishes.

**Definition 9.15.**

(a) Let $S$ be a surface and $K(p)$ its Gauss curvature at $p \in S$. We say that $p$ is

$$\begin{cases} elliptic & K(p) > 0 \\ hyperbolic & \text{if} \quad K(p) < 0 \\ flat & K(p) = 0 \end{cases}$$

The subset $\quad \begin{cases} \{\, p \in S \mid K(p) > 0 \,\} & elliptic \\ \{\, p \in S \mid K(p) < 0 \,\} & \text{is called} \quad hyperbolic \quad \text{region of } S \\ \{\, p \in S \mid K(p) = 0 \,\} & flat \end{cases}$

(b) Denote by $\kappa_1(p)$ and $\kappa_2(p)$ the principal curvatures at $p \in S$.

- We say that $p$ is *planar* if $\kappa_1(p) = 0$ and $\kappa_2(p) = 0$;
- we say that $p$ is *umbilic* if $\kappa_1(p) = \kappa_2(p)$.

**Example 9.16.**  (a) (Sphere) On a sphere $S^2(r)$, all points are elliptic and umbilic since both principal curvatures are $\kappa_1(p) = \kappa_2(p) = -1/r$. The converse is also true (see Theorem 9.19).

(b) (Plane) It is not hard to see that if $S$ is a plane (or an open subset of it) then all points of $S$ are planar. The converse is also true (see Theorem 9.19).

(c) (Hyperbolic paraboloid, Example 9.10) All points are hyperbolic (since $K(p) < 0$ for all $p \in S$), and in particular, there are no umbilic points or flat points.

(d) (Torus of revolution, Example 9.14) We have $K = 0$ iff $\cos(v/r) = 0$ i.e., if $v/r = \pi/2$ or $v/r = 3\pi/2$. This is the circle on top and bottom of the torus; this is the *flat region*. The *elliptic region* is given by points with $K > 0$, i.e., $-\pi/2 < v/r < -\pi/2$. The *hyperbolic region* is given by points with $K < 0$, i.e., $\pi/2 < v/r < 3\pi/2$.

There are no umbilic points on the torus of revolution: $|\kappa_1| < 1/r$, but $\kappa_2 = -1/r$, so the two principal curvatures cannot be the same. There are no planar points either $\kappa_2 = -1/r \neq 0$ everywhere).

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## 9.2 Some global theorems about curvature

**Theorem 9.17.** Every compact surface in $\mathbb{R}^3$ has at least one elliptic point.

**Remark 9.18.** The theorem is obviously false if either boundedness or closedness is dropped.

**Theorem 9.19.** Let $S$ be a surface in $\mathbb{R}^3$.

  (a) If all points of $S$ are umbilic and $K \neq 0$ in at least one point of $S$ then $S$ is a part of a sphere.

  (b) If all points of $S$ are planar then $S$ is part of a plane.

**Theorem 9.20** (Conjecture of Carathéodory)**.** Every compact surface in $\mathbb{R}^3$ (convex, homeomorphic to a sphere) has at least two umbilic points.

This theorem has recently (2008) been proved (with additional smoothness assumptions) by Brendan Guilfoyle and Wilhelm Klingenberg (Durham).

**Definition 9.21.** A surface $S$ is *minimal* if the mean curvature $H$ vanishes identically on $S$.

# 10  The Theorema Egregium of Gauss

"Theorema Egregium" means "Remarkable Theorem".

**Theorem 10.1** (Theorema Egregium)**.** The Gauss curvature of a surface in $\mathbb{R}^3$ depends on $E, F, G$ and their derivatives only (in a local parametrization).

In other words: the Gauss curvature is *intrinsic*.

**Corollary 10.2.** A local isometry preserves the Gauss curvature.

The converse is false: a map preserving the Gauss curvature is not necessarily a (local) isometry, see Remark 10.12.

**Remark 10.3.** Theorem 10.1 does *not* hold for the mean curvature: e.g. $H = 0$ (plane) but $H = 1/(2r)$ (cylinder), although the plane and the cylinder are locally isometric.

**Definition 10.4** (Christoffel symbols)**.** Let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization of a surface $S$ in $\mathbb{R}^3$. The Christoffel symbols $\Gamma_{ij}^k$ $(i, j, k \in \{1, 2\})$ are functions $\Gamma_{ij}^k\colon U \longrightarrow \mathbb{R}$ defined by

$$\boldsymbol{x}_{uu} = \Gamma_{11}^1 \boldsymbol{x}_u + \Gamma_{11}^2 \boldsymbol{x}_v + L\boldsymbol{N}$$
$$\boldsymbol{x}_{uv} = \Gamma_{12}^1 \boldsymbol{x}_u + \Gamma_{12}^2 \boldsymbol{x}_v + M\boldsymbol{N}$$
$$\boldsymbol{x}_{vu} = \Gamma_{21}^1 \boldsymbol{x}_u + \Gamma_{21}^2 \boldsymbol{x}_v + M\boldsymbol{N}$$
$$\boldsymbol{x}_{vv} = \Gamma_{22}^1 \boldsymbol{x}_u + \Gamma_{22}^2 \boldsymbol{x}_v + N\boldsymbol{N}$$

In particular, $\Gamma_{ij}^k = \Gamma_{ji}^k$.

**Lemma 10.5.**

  (a) We have the identities

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u = \frac{1}{2}E_u \qquad\qquad \boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v = \frac{1}{2}G_v$$

$$\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u = \frac{1}{2}E_v \qquad\qquad \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v = \frac{1}{2}G_u$$

$$\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u = F_v - \frac{1}{2}G_u \qquad\qquad \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v = F_u - \frac{1}{2}E_v$$

for the coefficients $E$, $F$ and $G$ of the first fundamental form with respect to a parametrization $\boldsymbol{x}$.

(b) The Christoffel symbols are uniquely determined by $E$, $F$, $G$ and their first derivatives.

**Corollary 10.6.** Gauss' Theorema Egregium implies that it is impossible to construct a distance preserving planar chart of the unite sphere.

**Corollary 10.7.** Gauss' Theorema Egregium allows us to define the Gauss curvature for *any* surface $S$ just using the *first fundamental form.*

**Example 10.8** (Gauss curvature of the hyperbolic plane)**.** Recall that we define the hyperbolic plane as a surface $\mathbb{H}$ parametrized by $x \colon U \longrightarrow H$ with

$$U = \mathbb{R} \times (0, \infty), \qquad E(u, v) = G(u, v) = \frac{1}{v^2}, \quad F(u, v) = 0.$$

*Step 1 — Christoffel symbols:* We first calculate the Christoffel symbols in the case that $F = 0$ (you can read off $\Gamma_{ij}^k$ directly):

$$\begin{cases} E\Gamma_{11}^1 = \frac{1}{2}E_u \\ G\Gamma_{11}^2 = -\frac{1}{2}E_v \end{cases} \qquad \begin{cases} E\Gamma_{12}^1 = \frac{1}{2}E_v \\ G\Gamma_{12}^2 = \frac{1}{2}G_u \end{cases} \qquad \begin{cases} E\Gamma_{22}^1 = -\frac{1}{2}G_u \\ G\Gamma_{22}^2 = \frac{1}{2}G_v \end{cases}$$

or in our case ($E$ and $G$ are functions of $v$ only).

$$\begin{cases} \frac{1}{v^2}\Gamma_{11}^1 = 0 \\ \frac{1}{v^2}\Gamma_{11}^2 = \frac{1}{v^3} \end{cases} \qquad \begin{cases} \frac{1}{v^2}\Gamma_{12}^1 = -\frac{1}{v^3} \\ \frac{1}{v^2}\Gamma_{12}^2 = 0 \end{cases} \qquad \begin{cases} \frac{1}{v^2}\Gamma_{22}^1 = 0 \\ \frac{1}{v^2}\Gamma_{22}^2 = -\frac{1}{v^3} \end{cases}$$

or

$$\begin{cases} \Gamma_{11}^1 = 0 \\ \Gamma_{11}^2 = \frac{1}{v} \end{cases} \qquad \begin{cases} \Gamma_{12}^1 = -\frac{1}{v} \\ \Gamma_{12}^2 = 0 \end{cases} \qquad \begin{cases} \Gamma_{22}^1 = 0 \\ \Gamma_{22}^2 = -\frac{1}{v}. \end{cases}$$

Therefore,

$$\boldsymbol{x}_{uu} = \Gamma_{11}^1\boldsymbol{x}_u + \Gamma_{11}^2\boldsymbol{x}_v + L\boldsymbol{N} = \frac{1}{v}\boldsymbol{x}_v + L\boldsymbol{N}$$

$$\boldsymbol{x}_{uv} = \Gamma_{12}^1\boldsymbol{x}_u + \Gamma_{12}^2\boldsymbol{x}_v + M\boldsymbol{N} = -\frac{1}{v}\boldsymbol{x}_u + M\boldsymbol{N}$$

$$\boldsymbol{x}_{vv} = \Gamma_{22}^1\boldsymbol{x}_u + \Gamma_{22}^2\boldsymbol{x}_v + N\boldsymbol{N} = -\frac{1}{v}\boldsymbol{x}_v + N\boldsymbol{N}$$

*Step 2 — Calculate $LN - M^2$:*

$$LN - M^2 = L\boldsymbol{N} \cdot N\boldsymbol{N} - M\boldsymbol{N} \cdot M\boldsymbol{N}$$

$$= (\boldsymbol{x}_{uu} - \frac{1}{v}\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} + \frac{1}{v}\boldsymbol{x}_v) - (\boldsymbol{x}_{uv} + \frac{1}{v}\boldsymbol{x}_u) \cdot (\boldsymbol{x}_{uv} + \frac{1}{v}\boldsymbol{x}_u)$$

$$= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} \quad -\frac{1}{v}\underbrace{\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v}_{=G_v/2=-1/v^3} +\frac{1}{v}\underbrace{\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v}_{=F_u-E_v/2=1/v^3} -\frac{1}{v^2}\underbrace{\boldsymbol{x}_v \cdot \boldsymbol{x}_v}_{=G=1/v^2}$$

$$-2\frac{1}{v}\underbrace{\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u}_{=E_v/2=-1/v^3} -\frac{1}{v^2}\underbrace{\boldsymbol{x}_u \cdot \boldsymbol{x}_u}_{=E=1/v^2}$$

$$= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} \quad +\frac{2}{v^4}.$$

We now have

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} = (\boldsymbol{x}_u \cdot \boldsymbol{x}_{vv})_u - (\boldsymbol{x}_u \cdot \boldsymbol{x}_{uv})_v$$

$$= (F_v - \frac{1}{2}G_u)_u - \frac{1}{2}E_{vv} = -\frac{\partial^2}{\partial v^2}\frac{1}{2v^2} = -\frac{3}{v^4}.$$

*Step 3 — Calculate K:* Since $EG - F^2 = 1/v^4$, we have finally

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-3/v^4 + 2/v^4}{1/v^4} = -1.$$

As a result, we have: the hyperbolic plane has constant curvature $-1$.

**Remark 10.9.**

(a) In Example 10.8 (or more generally, in all examples where we calculate the Gauss curvature from $E$, $F$ and $G$ only) we used the fact that $S \subset \mathbb{R}^3$ (at least locally), because we used the formulae for $\boldsymbol{x}_{uu}$ etc. involving the normal vector $\boldsymbol{N}$. This is for convenience only, to remember the procedure. More precisely, we should use the formula

$$K = \left(\frac{LN - M^2}{EG - F^2} =\right) \frac{E_{vv}/2 + F_{uv} - E_{vv}/2 + \text{terms in } E,F,G \text{ and derivatives}}{EG - F^2}$$

as *the definition* of $K$ for a general surface as we did in Theorem 10.1.

(b) Recall that for plane curves the signed curvature defined a curve up to an isometry of the plane. What about a similar result for surfaces? Does the Gauss curvature define a surface uniquely (or up to what data the surface is unique)?

The answer to the uniqueness is *negative*, as Remark 10.12 shows: there exist surfaces $S$, $\widetilde{S}$ and a diffeomorphism $f \colon S \longrightarrow \widetilde{S}$ ($f$ is bijective, smooth and $f^{-1}$ is also smooth) which is *not* an isometry, but for which the Gauss curvature is preserved (i.e., $K(p) = \widetilde{K}(f(p))$, if $K$ resp. $\widetilde{K}$ is the Gauss curvature of $S$ resp. $\widetilde{S}$).

**Example 10.10.** (Gauss curvature in an orthogonal parametrization).
In an orthogonal parametrization ($F = 0$) we have

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

**Example 10.11.** (Flat torus in $\mathbb{R}^4$).
Let $T = S^1 \times S^1 \subset \mathbb{R}^4$ be the so-called *flat torus*. We have a standard parametrization

$$\boldsymbol{x}(u,v) = (\cos u, \sin u, \cos v, \sin v), \qquad (u,v) \in U$$

with $U = (0, 2\pi) \times (0, 2\pi)$ (and other suitable sets to cover all of $S$).
We have

$$\boldsymbol{x}_u = (-\sin u, \cos u, 0, 0) \qquad \text{and} \qquad \boldsymbol{x}_v = (0, 0, -\sin v, \cos v),$$

so that $E = G = 1$ and $F = 0$.
Therefore the Gauss curvature is

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right) = 0.$$

**Example 10.12.** (Surfaces with the same Gauss curvature are not necessarily isometric).

Let $U = (0, 2\pi) \times (0, \infty)$ and let $S, \widetilde{S}$ be the surfaces defined by $S = \boldsymbol{x}(U)$, $\widetilde{S} = \boldsymbol{y}(U)$, where $\boldsymbol{x}, \boldsymbol{y} \colon U \longrightarrow \mathbb{R}^3$ are defined by

$$\boldsymbol{x}(u, v) = (v \cos u, v \sin u, u), \quad \boldsymbol{y}(u, v) = (v \cos u, v \sin u, \log v), \quad (u, v) \in U.$$

(thus $S$ is an open subset of the helicoid and $\widetilde{S}$ is an open subset of a surface of revolution).

The coefficients of the first fundamental forms of $S$ resp. $\widetilde{S}$ w.r.t. $\boldsymbol{x}$ resp. $\boldsymbol{y}$ are

$$E = v^2 + 1, \quad F = 0, \quad G = 1 \qquad \text{and} \qquad \widetilde{E} = v^2, \quad \widetilde{F} = 0, \quad \widetilde{G} = 1 + \frac{1}{v^2}.$$

Calculating the Gauss curvature for $S$ and $\widetilde{S}$ gives

$$K(\boldsymbol{x}(u, v)) = \widetilde{K}(\boldsymbol{y}(u, v)) = -\frac{1}{(v^2 + 1)^2},$$

and hence $K(p) = \widetilde{K}(f(p))$.

Since the coefficients of the first fundamental form $S$ and $\widetilde{S}$ are different, $f$ cannot be a local isometry (note that $f \circ \boldsymbol{x} = \boldsymbol{y}$, so that $(f \circ \boldsymbol{x})_u \cdot (f \circ \boldsymbol{x})_u = \boldsymbol{y}_u \cdot \boldsymbol{y}_u = \widetilde{E}$ etc.), so since $E \neq \widetilde{E}$, $f$ cannot be an isometry by Proposition 8.15.

**Example 10.13.** Suppose that $E = 1 = G$ and $F = \cos \vartheta$, where $\vartheta$ is the angle between the coordinate curves. Then $K = \frac{\vartheta_{uv}}{\sin \vartheta}$. (Compare to HW 15.2)

# 11   Curves on surfaces

## 11.1   Coordinate curves

**Definition 11.1.** Let $S$ be a regular surface in $\mathbb{R}^n$. A *curve on the surface $S$* is a smooth map $\boldsymbol{\alpha} \colon I \longrightarrow S$ ($I \subset \mathbb{R}$ is an interval).

**Remark 11.2.** Recall: If $\boldsymbol{x} \colon U \longrightarrow S$ is a local parametrisation ($U \subset \mathbb{R}^2$ open) and $\boldsymbol{\alpha} \colon I \longrightarrow \boldsymbol{x}(U)$ a curve in $\boldsymbol{x}(U) \subset U$, then we can write

$$\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s)),$$

and

$$\boldsymbol{\alpha}' = u' \boldsymbol{x}_u + v' \boldsymbol{x}_v,$$

which implies

$$\|\boldsymbol{\alpha}'(t)\| = \sqrt{E(u(t), v(t))u'(t)^2 + 2F(u(t), v(t))u'(t)v'(t) + \dots}$$

**Example 11.3. Coordinate curves:** Let $\boldsymbol{x} \colon U \longrightarrow S$ be a local parametrization ($U \subset \mathbb{R}^2$ open) and $(u_0, v_0) \in U$, then

$$u \mapsto \boldsymbol{x}(u, v_0)$$
$$v \mapsto \boldsymbol{x}(u_0, v)$$

are called *coordinate curves* through $p = \boldsymbol{x}(u_0, v_0)$. The local parametrization is given by $(u(s), v(s)) = (s, v_0)$ for the first, and $(u(s), v(s)) = (u_0, s)$ for the second.

One should note that coordinate curves are not intrinsic, they depend on the parametrization.

## 11.2 Geodesic and normal curvature

Assume now that $S \subset \mathbb{R}^3$, $\boldsymbol{\alpha} \colon I \longrightarrow S \subset \mathbb{R}^3$ is a unit speed curve. Then $\boldsymbol{\alpha}'(s)$ and $\boldsymbol{\alpha}''(s)$ are orthogonal, and

$$\|\boldsymbol{\alpha}''(s)\| = \kappa(s),$$

where $\kappa(s)$ denotes the *curvature* of $\boldsymbol{\alpha}$ as a space curve.

Denote by $\boldsymbol{N}(\boldsymbol{\alpha}(s))$ the Gauss map of the surface $S$ at $\boldsymbol{\alpha}(s)$. Since $\boldsymbol{\alpha}''$ is orthonormal to $\boldsymbol{\alpha}'$, it lies in the plane spanned by $\boldsymbol{N}$ and $\boldsymbol{N} \times \boldsymbol{\alpha}'$.

**Definition 11.4** (Geodesic and normal curvature). If $\boldsymbol{\alpha} \colon I \longrightarrow S$ is a curve on a surface $S$ (with Gauss map $\boldsymbol{N}$) parametrized by arc lenth, then we can write

$$\boldsymbol{\alpha}''(s) = \kappa_{\mathrm{g}}(s)\boldsymbol{N}(\boldsymbol{\alpha}(s)) \times \boldsymbol{\alpha}'(s) + \kappa_{\mathrm{n}}(s)\boldsymbol{N}(\boldsymbol{\alpha}(s)).$$

We call $\kappa_{\mathrm{g}} \colon I \longrightarrow \mathbb{R}$ the *geodesic curvature* and $\kappa_{\mathrm{n}} \colon I \longrightarrow \mathbb{R}$ the *normal curvature* of $\boldsymbol{\alpha}$ in $S$.

For a curve with an arbitrary parametrization on $S$ the geodesic and normal curvatures are defined to be the same as for its unit speed reparametrization, i.e. if $\boldsymbol{\beta} \colon J \to S$ is a curve, $\boldsymbol{\alpha} \colon I \to S$ is a unit speed curve, and $\boldsymbol{\beta}(t(s)) = \boldsymbol{\alpha}(s)$, then $\kappa_{\boldsymbol{\beta},\mathrm{n}}(t(s)) = \kappa_{\boldsymbol{\alpha},\mathrm{n}}(s)$, and $\kappa_{\boldsymbol{\beta},\mathrm{g}}(t(s)) = \kappa_{\boldsymbol{\alpha},\mathrm{g}}(s)$. In other words, normal and geodesic curvatures are invariant under reparametrizations by definition.

**Remark 11.5.** We have (if $\boldsymbol{\alpha}$ is parametrized by arc length!)

$$\kappa_{\mathrm{n}} = \boldsymbol{\alpha}'' \cdot \boldsymbol{N} \qquad \text{and} \qquad \kappa_{\mathrm{g}} = \boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \times \boldsymbol{\alpha}')$$

Furthermore, recall that the curvature $\kappa$ of a *space curve* is given by $\kappa = \|\boldsymbol{\alpha}''\|$ (if $\boldsymbol{\alpha}$ is parametrized by arc length), and since $\boldsymbol{N}$ and $\boldsymbol{N} \times \boldsymbol{\alpha}'$ form an orthonormal system, we have by Pythagoras' Theorem

$$\kappa = \|\boldsymbol{\alpha}''\| = \sqrt{\kappa_{\mathrm{g}}^2 + \kappa_{\mathrm{n}}^2}$$

**Example 11.6.**    (a) (Plane).

$S = \{\, (u, v, 0) \mid (u, v) \in \mathbb{R}^2 \,\}$, then $\boldsymbol{N} = (0, 0, 1)$.

Let $\boldsymbol{\alpha} \colon I \longrightarrow S$, $\boldsymbol{\alpha}(s) = (u(s), v(s), 0)$, parametrized by arclength; then $\boldsymbol{\alpha}' = (u', v', 0)$, $\boldsymbol{n} \times \boldsymbol{\alpha}' = (-v', u', 0)$ so that

$$\boldsymbol{\alpha}'' = (u'', v'', 0) = \kappa_{\mathrm{g}}(\boldsymbol{N} \times \boldsymbol{\alpha}') + \kappa_{\mathrm{n}}\boldsymbol{N} = \kappa_{\mathrm{g}}(-v', u', 0) + \kappa_{\mathrm{n}}(0, 0, 1)$$

so that $\kappa_{\mathrm{n}} = 0$, and, if $\kappa$ is the curvature of $\boldsymbol{\alpha}$, $\kappa = \kappa_{\mathrm{g}}$ (if $\boldsymbol{\alpha}$ is considered as a plane curve) or $\kappa = |\kappa_{\mathrm{g}}|$ (if $\boldsymbol{\alpha}$ is considered as a space curve).

(b) (Lines on surfaces).

Assume that $\boldsymbol{\alpha}(s) = p + s\boldsymbol{v}$, $\|v\| = 1$, parametrizes a line ($s \in I \subset \mathbb{R}$) and that $\boldsymbol{\alpha}(s) \in S$ for all $s \in I$ for some surface $S \subset \mathbb{R}^3$. Then

$$\boldsymbol{\alpha}' = \boldsymbol{v}, \qquad \boldsymbol{\alpha}'' = (0, 0, 0),$$

so that $\kappa_{\mathrm{g}} = 0$ and $\kappa_{\mathrm{n}} = 0$, i.e., the geodesic and normal curvature of a line on a surface both vanish.

**Theorem 11.7** (Meusnier). All curves $\boldsymbol{\beta}$ through $p \in S$ with the same tangent vector $\boldsymbol{w} \in T_pS$ have the same normal curvature

$$\kappa_{\mathrm{n}}(s) = I\!I_p\Big(\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}\Big).$$

In particular, the value $\kappa_{\mathrm{n}}(\boldsymbol{w})$ is well defined for any $\boldsymbol{w} \in T_pS$.

**Corollary.** Let $p \in S$, $\boldsymbol{w} \in T_p S$, and let $\Pi$ be the plane through $p$ spanned by $\boldsymbol{N}(p)$ and $\boldsymbol{w}$. Then $\kappa_n(\boldsymbol{w}) = \kappa(\Pi \cap S)$, where $\Pi \cap S$ is considered as a plane curve with tangent vector $\boldsymbol{w}$ at $p$.

**Proposition 11.8.** (Normal curvature in a local parametrization)

Let $S$ be a surface in $\mathbb{R}^3$, and let $E, F, G$ and $L, M, N$ be the coefficient of the first and second fundamental forms respectively w.r.t. a parametrization $\boldsymbol{x}$. Further, let $\boldsymbol{\alpha}$ be a curve in $S$ (not necessarily parametrized by arc length) with local parametrization $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$. Then

$$\kappa_n = II_p\left(\frac{\boldsymbol{\alpha}'}{\|\boldsymbol{\alpha}'\|}\right) = \frac{(u')^2 L + 2u'v'M + (v')^2 N}{(u')^2 E + 2u'v'F + (v')^2 G} = \frac{II_p(\boldsymbol{\alpha}')}{I_p(\boldsymbol{\alpha}')}$$

**Proposition 11.9.** Let $\boldsymbol{\beta} \colon I \longrightarrow S$ be a curve not necessarily parametrized by arc length, and let $\boldsymbol{N}$ be the Gauss map of $S$. Then the geodesic curvature of $\boldsymbol{\beta}$ can be calculated as

$$\kappa_g = \frac{1}{\|\boldsymbol{\beta}'\|^3}(\boldsymbol{\beta}' \times \boldsymbol{\beta}'') \cdot \boldsymbol{N}.$$

**Definition 11.10.** (Asymptotic curves) A curve $\boldsymbol{\alpha}$ on a surface $S \subset \mathbb{R}^3$ is called an *asymptotic curve* if its normal curvature vanishes identically (i.e., if $\kappa_n = 0$).

**Remark 11.11.**    (i) The following are equivalent (TFAE):

(a) $\boldsymbol{\alpha}$ is an asymptotic curve;

(b) $\boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \circ \boldsymbol{\alpha}) = 0$ (if $\boldsymbol{N}$ is the Gauss map of $S$ and $\boldsymbol{\alpha}$ is parametrized by arc length);

(c) $\kappa_n = 0$;

(d) $II_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s)) = 0$ for all $s$ ($\boldsymbol{\alpha}$ not necessarily parametrized by arc length);

(e) $(u')^2 L + 2u'v'M + (v')^2 N = 0$ in a local parametrization $s \mapsto \boldsymbol{x}(u(s), v(s))$ of $\boldsymbol{\alpha}$.

In particular, $II_p$ is not positive or negative definite along $\boldsymbol{\alpha}$, so $\boldsymbol{\alpha}$ has to be in the *hyperbolic* or *flat* region of the surface.

(ii) $\kappa_n(\boldsymbol{w}) = 0$ for $\boldsymbol{w} \in T_p S$ implies $K(p) \leq 0$.

(iii) If $\boldsymbol{\alpha}$ is a line on $S$, then $\kappa_n = 0$, i.e., any line on a surface is an asymptotic curve.

## 11.3    Asymptotic curves

**Example 11.12.** (Asymptotic curves on a surface of revolution/catenoid)

Recall that on a surface of revolution obtained by rotating a curve $\boldsymbol{\alpha}$ given by $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$ around the $z$-axis, we have

$$L = \frac{-fg'}{\|\boldsymbol{\alpha}'\|}, \quad M = 0, \quad N = \frac{f''g' - f'g''}{\|\boldsymbol{\alpha}'\|}$$

(see Example 9.13). A curve $\boldsymbol{\beta}$ parametrized locally by $\boldsymbol{\beta}(t) = \boldsymbol{x}(u(t), v(t))$ is an asymptotic curve iff $(u')^2 L + 2u'v'M + (v')^2 N = 0$, i.e., iff

$$(u')^2 fg' = (v')^2 (f''g' - f'g'')$$

If in particular, $f(v) = \cosh v$ and $g(v) = v$ (i.e., the surface of revolution is a *catenoid*), then the above equation becomes

$$(u')^2 \cosh v = (v')^2 \cosh v, \quad \text{or,} \quad u' = \pm v', \quad \text{i.e.,} \quad u = \pm v + c$$

for some constant $c \in \mathbb{R}$.

## 11.4   Lines of curvature

**Definition 11.13.** (Lines of curvature)
   A curve $\boldsymbol{\alpha}\colon I \longrightarrow S$ on a surface $S$ in $\mathbb{R}^3$ is called a *line of curvature* if $\boldsymbol{\alpha}'(s)$ is a principal direction at $\boldsymbol{\alpha}(s)$ for all $s \in I$, i.e., $\boldsymbol{\alpha}'(s)$ is an eigenvector of the Weingarten map at $\boldsymbol{\alpha}(s)$ for all $s$.
   Equivalently, $\boldsymbol{\alpha}$ is a line of curvature if there is a function $\lambda\colon I \longrightarrow \mathbb{R}$ such that

$$-d\boldsymbol{N}_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s)) = \lambda(s)\boldsymbol{\alpha}'(s)$$

for all $s \in I$. (Here $\lambda(s)$ is a principal curvature at $\boldsymbol{\alpha}(s)$.)

**Remark 11.14.** Note that if the eigenvalues of a symmetric $2 \times 2$-matrix are different, then the corresponding eigenvectors are orthogonal. Hence, each non-umbilic point $(\kappa_1(p) \neq \kappa_2(p))$ has two lines of curvature through it, and they intersect orthogonally. In an umbilic point, this family of orthogonally intersecting curves has a singularity.
   Moreover any direction at an umbilic point is principal. In particular, on a sphere $(\kappa_1 = \kappa_2 > 0)$ or a plane $(\kappa_1 = \kappa_2 = 0)$ any curve is a line of curvature.

**Proposition 11.15.** (Lines of curvature in a local parametrisation) Let $E, F, G$ and $L, M, N$ be the coefficients of the first and second fundamental forms respectively w.r.t. a local parametrization $\boldsymbol{x}\colon U \longrightarrow S$, and let $\boldsymbol{\alpha}$ be a curve in $S$ with local parametrization $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$. Then $\boldsymbol{\alpha}$ is a line of curvature if and only if

$$\det \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ L & M & N \end{pmatrix} = 0$$

or, equivalently,

$$(FN - GM)(v')^2 + (EN - GL)u'v' + (EM - FL)(u')^2 = 0.$$

**Example 11.16.** (Hyperbolic paraboloid)
   Let $S = \{\, (x, y, z) \mid xy = z \,\}$ be a hyperbolic paraboloid parametrized by $\boldsymbol{x}(u, v) = (u, v, uv)$. Then

$$\boldsymbol{x}_u = (1, 0, v), \quad \boldsymbol{x}_v = (0, 1, u), \quad \boldsymbol{N} = D^{-1}(-v, -u, 1), \quad D = (u^2 + v^2 + 1)^{1/2}$$
$$\boldsymbol{x}_{uu} = (0, 0, 0), \quad \boldsymbol{x}_{uv} = (0, 0, 1), \quad \boldsymbol{x}_{vv} = (0, 0, 0)$$

and

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u = 1 + v^2, \quad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = uv, \quad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v = 1 + u^2,$$
$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = 0, \quad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = 1/D, \quad N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = 0$$

   Therefore, $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ is a *line of curvature* iff

$$\det \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ 1 + v^2 & uv & 1 + u^2 \\ 0 & 1/D & 0 \end{pmatrix} = (u')^2(1 + v^2)/D - (v')^2(1 + u^2)/D = 0,$$

which is equivalent to

$$\frac{u'}{(1 + u^2)^{1/2}} = \pm \frac{v'}{(1 + v^2)^{1/2}},$$

and after integrating,

$$\operatorname{arcsinh} u = \pm \operatorname{arcsinh} v + c$$

for some constant $c \in \mathbb{R}$. For example, if $c = 0$, then $u = \pm v$, or $s \mapsto \boldsymbol{x}(s, \pm s) = (s, \pm s, \pm s^2)$ are the lines of curvature through $p = (0, 0, 0)$.

The *asymptotic curves* here are given by

$$(u')^2 L + 2u'v'M + (v')^2 N = 2u'v'/D = 0,$$

i.e., $u' = 0$ or $v' = 0$, so the asymptotic curves are the coordinate curves $s \mapsto \boldsymbol{x}(s, v_0)$ or $s \mapsto \boldsymbol{x}(u_0, s)$

**Remark 11.17.** (a) On a *line of curvature*, the *normal curvature* is a *principal curvature*.

Indeed, since $\boldsymbol{\alpha}$ is a line of curvature, we have $-d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s)) = \lambda(s)\boldsymbol{\alpha}'(s)$, and $\lambda(s)$ is a principal curvature at $\boldsymbol{\alpha}(s)$.

On the other hand,

$$\kappa_{\mathrm{n}}(s) = \frac{II_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s))}{I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s)} = \frac{\langle -d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s)), \boldsymbol{\alpha}'(s)\rangle}{\langle \boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle} = \frac{\langle \lambda(s)\boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle}{\langle \boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle} = \lambda(s)$$

(b) Assume that a line $\boldsymbol{\alpha}$ (or a part of it) belongs to a surface. When is this line a *line of curvature*?

On a line, the normal curvature is 0, hence by the first part, one of its principal curvatures, say $\kappa_1$, has to vanish on $\boldsymbol{\alpha}$. But this means that the Gauss curvature (as the product of the two principal curvatures $K = \kappa_1\kappa_2$) has to vanish (and vice versa). Hence if $\boldsymbol{\alpha}: I \longrightarrow S$ is a line in $S$, then

$$\boldsymbol{\alpha} \text{ is a line of curvature} \quad \Leftrightarrow \quad \big(K(\boldsymbol{\alpha}(s)) = 0 \quad \forall s \in I\big).$$

This is equivalent to $LN - M^2 = 0$.

**Proposition 11.18.** (Lines of curvature for a principal parametrization)

If $\boldsymbol{x}$ is a principal parametrization of a surface $S \subset \mathbb{R}^3$ (i.e., $F = 0$ and $M = 0$), then the coordinate curves are lines of curvature.

**Example 11.19.** (Lines of curvature for a surface of revolution)

On a surface of revolution, the coordinate curves of the standard parametrization given by $\boldsymbol{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$ are also lines of curvature.

**Remark 11.20.** Note that the converse of Proposition 11.18 is also true in the following sense: if a parametrization $\boldsymbol{x}$ is principal and the umbilic points are isolated, then the lines of curvature are coordinate curves.

## 12  Geodesics

**Definition 12.1.** Let $\boldsymbol{\alpha}: I \longrightarrow S$ be a (regular) curve on a surface $S \subset \mathbb{R}^3$. $\boldsymbol{\alpha}$ is called *geodesic* if $\boldsymbol{\alpha}''$ is normal to $S$ (i.e., $\boldsymbol{\alpha}''(s)$ is orthogonal to $T_{\boldsymbol{\alpha}(s)}S$ for all $s \in I$).

Note that the curve does not need to be parametrized by arc length, but we have:

**Proposition 12.2** (Geodesics have constant speed). Let $\boldsymbol{\alpha}$ be a geodesic, then $\|\boldsymbol{\alpha}'\|$ is constant, i.e., there exists $c > 0$ such that $\boldsymbol{\alpha}'(s) = c$ for all $s \in I$.

In other words, a geodesic is always parametrized *proportionally* to arc length.

**Example 12.3.**

(a) **Lines are geodesics.**

Let $S$ be a surface and $\boldsymbol{\alpha}$ be a line in $S$. Then $\boldsymbol{\alpha}''(s) = 0$, hence $\boldsymbol{\alpha}''$ is normal to any vector (in particular to the tangent plane $T_{\boldsymbol{\alpha}(s)}S$). Therefore, $\boldsymbol{\alpha}$ is a geodesic.

(b) **Geodesics on a cylinder.**

Let $S = \{\, (x, y, z) \mid x^2 + y^2 = 1 \,\}$, then any geodesic $\boldsymbol{\alpha}$ on $S$ is parametrized by

$$\boldsymbol{\alpha}(s) = (\cos(as + b), \sin(as + b), \lambda s + \mu)$$

for some $\lambda, \mu, a, b \in \mathbb{R}$. If $a = 0$ then $\boldsymbol{\alpha}$ is a meridian, if $\lambda = 0$ then $\boldsymbol{\alpha}$ is a parallel.

(c) **Great circles on a sphere are geodesics.**

A *great circle* on a sphere is the curve given by the intersection of the sphere with a plane through its origin.

Let $S = \{\, (x, y, z) \mid x^2 + y^2 + z^2 = 1 \,\}$, and let $\boldsymbol{v}, \boldsymbol{w}$ be orthonormal in $\mathbb{R}^3$. Set

$$\boldsymbol{\alpha}(s) = (\cos s)\boldsymbol{v} + (\sin s)\boldsymbol{w}$$

for $s \in I$ ($I$ some interval). Then $\boldsymbol{\alpha}''(s) = -\boldsymbol{\alpha}(s) = -\boldsymbol{N}(\boldsymbol{\alpha}(s))$, hence $\boldsymbol{\alpha}$ is orthogonal to $T_{\boldsymbol{\alpha}(s)}S$ and $\boldsymbol{\alpha}$ is a geodesic.

**Proposition 12.4** (Equivalent characterization of geodesics)**.** The following are equivalent (TFAE):

(a) $\boldsymbol{\alpha}$ is a geodesic;

(b) $\boldsymbol{\alpha}$ has constant speed and its geodesic curvature vanishes.

**Proposition 12.5** (Geodesics in a local parametrization)**.** Let $\boldsymbol{\alpha} \colon I \longrightarrow S$ be a curve on a surface $S \subset \mathbb{R}^3$, and let $\boldsymbol{x} \colon U \longrightarrow S$ be a local parametrization. We write $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ and $E, F, G$ for the coefficients of the first fundamental form w.r.t. $\boldsymbol{x}$. Then the following are equivalent:

(a) $\boldsymbol{\alpha}$ is a geodesic;

(b) $\boldsymbol{\alpha}'' \cdot \boldsymbol{x}_u = 0$ and $\boldsymbol{\alpha}'' \cdot \boldsymbol{x}_v = 0$;

(c)

$$u''E + \frac{1}{2}(u')^2 E_u + u'v'E_v + (v')^2\left(F_v - \frac{1}{2}G_u\right) + v''F = 0,$$

$$v''G + \frac{1}{2}(v')^2 G_v + u'v'G_u + (u')^2\left(F_u - \frac{1}{2}E_v\right) + u''F = 0.$$

Let us now state the main theorem about geodesics:

**Theorem 12.6** (Local existence and uniqueness of geodesics)**.** (a) Let $p \in S$, $\boldsymbol{w} \in T_pS \setminus \{0\}$. Then there exists $\varepsilon > 0$ and a *unique* geodesic $\boldsymbol{\alpha} \colon (-\varepsilon, \varepsilon) \longrightarrow S$ such that $\boldsymbol{\alpha}(0) = p$ and $\boldsymbol{\alpha}'(0) = \boldsymbol{w}$.

(b) Geodesics are determined entirely by the coefficients of the first fundamental form $E$, $F$ and $G$ (and their derivatives) in a local parametrization.

**Corollary 12.7** (Isometries take geodesics to geodesics)**.** Let $f \colon S \longrightarrow \widetilde{S}$ be a local isometry between two surfaces $S$ and $\widetilde{S}$, and let $\boldsymbol{\alpha} \colon I \longrightarrow S$ be a geodesic on $S$. Then $f \circ \boldsymbol{\alpha} \colon I \longrightarrow \widetilde{S}$ is also a geodesic on $\widetilde{S}$.

**Example 12.8.**

(a) **Plane.**

We know that $E = G = 1$ and $F = 0$ (in the standard parametrization $(u, v) \in \mathbb{R}^2$), so the local equation for a geodesic is

$$u'' = 0 \qquad \text{and} \qquad v'' = 0$$

This means that

$$u(s) = u_0 + as \qquad \text{and} \qquad v(s) = v_0 + bs$$

for some numbers $u_0, v_0, a, b$ ($(u_0, v_0)$ is the starting point and $\boldsymbol{w} = (a, b)$ is the initial speed vector). These are all geodesics on a plane

(b) **Cylinder.**

Let $S := \{ (x, y, z) \mid x^2 + y^2 = 1 \}$ be a cylinder and $f \colon \mathbb{R}^2 \longrightarrow S$ be given by $f(u, v) = (\cos u, \sin u, v)$, then $f$ is a local isometry. Geodesics on the cylinder $S$ are just images of lines under $f$:

- lines $s \mapsto (\cos u_0, \sin u_0, s)$ ($u_0$ some constant): image of the line $s \mapsto (u_0, s)$;

- circles $s \mapsto (\cos s, \sin s, v_0)$ ($v_0$ some constant): image of the line $s \mapsto (s, v_0)$;

- helices $s \mapsto (\cos s, \sin s, v_0 + as)$ ($v_0$, $a$ some constants): image of the line $s \mapsto (s, v_0 + as)$ (the circles above are the case $a = 0$)

These are all geodesics (use the local *uniqueness* result of Theorem 12.6), cf. Example 12.3.

**Remark 12.9** (Minimizing property of geodesics). (a) The shortest curve between two points on a surface is a geodesic (if parametrized proportionally to arc length).

(b) Converse is false: not all geodesics connecting two points minimize the distance.

(c) A minimizing curve (a geodesic) might not be unique. Moreover, there might be infinitely many of these.

(d) There might be no geodesic joining two points on a surface.

**Example 12.10** (Geodesics on a surface of revolution). Let $S$ be a surface of revolution with local parametrization

$$\boldsymbol{x}(u, v) = (f(v) \cos u, f(v) \sin u, g(v)),$$

and let $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ be a geodesic on $S$. Then the equations from Prop. 12.5 reduce to

$$u'' E + u' v' E_v = 0,$$

$$v'' G + \frac{1}{2} v'^2 G_v - \frac{1}{2} u'^2 E_v = 0.$$

The first equation is equivalent to $(u' E)' = 0$, or

$$u' = \frac{c}{f^2}$$

for some constant $c \in \mathbb{R}$.

Assuming that the the generating curve $(f, 0, g)$ is unit speed, the second equation is reduced to $v'' G - u'^2 E_v / 2 = 0$, or, equivalently,

$$v'' - u'^2 f f' = 0$$

as $E = f^2$.

**Corollary.**  (a) All meridians are geodesics

(b) A parallel $v = v_0$ is geodesic if and only if $f'(v_0) = 0$.

**Proposition 12.11** (Clairaut relation). Let $S$ be a surface of revolution with local parametrization

$$\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)),$$

and let $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ be a geodesic on $S$. Denote by $\Theta(s)$ the angle formed by $\boldsymbol{\alpha}'(s)$ and the parallel through $\boldsymbol{\alpha}(s)$. Then

$$f(v(s))\cos\Theta(s) = \text{const}$$

**Example 12.12** (Torus of revolution). Let $S$ be a torus of revolution with local parametrization

$$\boldsymbol{x}(u,v) = ((R + r\cos v)\cos u, (R + r\cos v)\sin u, r\sin v)$$

for $0 < r < R$. Let $\boldsymbol{\alpha}(s)$ be a geodesic on $S$ through a point $\boldsymbol{\alpha}(0) = (R+r, 0, 0)$. Denote by $\Theta_0$ the angle formed by $\boldsymbol{\alpha}'(0)$ and $\boldsymbol{x}_u$. Then $\boldsymbol{\alpha}(s)$ satisfies the equation

$$(R + r\cos v(s))\cos\Theta(s) = (R + r)\cos\Theta_0$$

**Definition 12.13.** A geodesic $\boldsymbol{\alpha}\colon I \longrightarrow S$ is *closed* if there is $c \in \mathbb{R}_+$ such that $\boldsymbol{\alpha}(s+c) = \boldsymbol{\alpha}(s)$ for every $s \in I$.

**Example 12.14.**  (a) Every geodesic on a sphere is closed.

(b) The only closed geodesics on a cylinder are parallels.

**Example 12.15.** There are no closed geodesics on an elliptic paraboloid of revolution.

**Example 12.16.** (Geodesics on hyperbolic plane).  Geodesics in the upper half-plane with the first fundamental form given by $E = G = 1/v^2$, $F = 0$ are vertical rays and semi-circles orthogonal to the boundary.

(To prove this, we first show that the vertical rays are geodesic by existence and uniqueness, then we apply isometries of $\mathbb{H}^2$ to obtain all the other geodesics - we know, we obtain all, again, by uniqueness).

# 13 Gauss–Bonnet theorems

## 13.1 A bit of topology

**Definition 13.1.**  (a) A surface $S \subset \mathbb{R}^n$ is a *closed surface* if $S$ is bounded, connected and closed (as a set).

(b) A surface is *oriented* if the Gauss map can be defined globally as a continuous map.

(c) A *region* of a surface $S$ is a subset of $S$ such that its boundary consists of a finite number of smooth curves (called *edges*) and its interior is non-empty. We call the points in which two smooth curves meet on the boundary *vertices* (and we assume for simplicity that the curves meet non-tangentially).

(d) A *triangle* is a region with three edges and three vertices homeomorphic to a disc (note that the edges, as well as the vertices, may coincide).

(e) A *triangulation* of a (bounded) region $R$ is a subdivision of $S$ into a finite number of triangles meeting only in common edges or common vertices.

(f) The *Euler characteristic* of a region $R$ is defined by

$$\chi(R) := F(R) - E(R) + V(R)$$
$$= \#\text{triangles} - \#\text{edges} + \#\text{vertices},$$

where $F(R)$ is the number of triangles, $E(R)$ the number of edges and $V(R)$ the number of vertices of the triangulation.

**Example 13.2.** A closed disc has Euler characteristic 1, a sphere has Euler characteristic 2, a closed cylinder $S^1 \times [0,1]$ (as well as a torus) has Euler characterisic 0.

*A priori*, the Euler characteristic may depend on the triangulation.

**Theorem 13.3.** The Euler characteristic is independent of the triangulation.

Basically, oriented closed surfaces can be topologically characterized by their Euler characteristic:

$$\chi(S) = 2 - 2g,$$

where $g$ is the *genus* of $S$ (roughly, the number of "handles" in $S$).

**Theorem 13.4** (Jordan Curve Theorem)**.** Let $S$ be a surface homeomorphic to the plane, and let $\alpha \colon [0,1] \longrightarrow S$ be a simple closed curve (i.e., $\alpha(0) = \alpha(1)$ and $\alpha(t_1) \neq \alpha(t_2)$ for $t_1 < t_2$ other than $t_1 = 0$, $t_2 = 1$). Then $S \setminus \alpha(I)$ has exactly two components, and one of them is homeomorphic to a disc.

## 13.2   The Gauss–Bonnet theorem

**Definition 13.5.** Let $R \subset S$ be a region.

(a) Denote by $\mathrm{d}A$ the area measure of a surface $S$ (locally, $\mathrm{d}A = \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v$), and we will write

$$\int_R K\,\mathrm{d}A$$

for the integral of the Gauss curvature over $R$ (the *total* Gauss curvature of $R$).

(b) Denote by $\mathrm{d}s$ the length measure of a curve or the boundary of a region, we will write

$$\int_{\partial R} \kappa_\mathrm{g}\,\mathrm{d}s = \sum_{j=1}^{r} \int_{I_j} \kappa_{\mathrm{g},\alpha_j}(s)\,\mathrm{d}s_j$$

for the line integral of the geodesic curvature along the boundary of a region consisting of $r$ smooth curves $\alpha_j$.

(c) Let us parametrize the curves along $\partial R$ counter-clockwise, and the curves are numbered in the same direction. We define the *angle* $\vartheta_j$ at the vertex $v_j$ (where curve $\alpha_{j-1}$ and $\alpha_j$ meet) as the angle between the tangent vector of $\alpha_{j-1}$ with the tangent vector of $\alpha_j$, i.e. $\vartheta_j$ is the exterior angle of $R$ at $v_j$.

Note that all objects here are intrinsic (Gauss curvature, geodesic curvature), so we can state the Gauss–Bonnet Theorem for any surface $S$ embedded in $\mathbb{R}^n$ (not only for $n = 3$).

**Theorem 13.6** (Global Gauss–Bonnet Theorem)**.** Let $R$ be a region in an oriented surface $S$. Then

$$\int_R K \, \mathrm{d}A + \int_{\partial R} \kappa_g \, \mathrm{d}s + \sum_{j=1}^r \vartheta_j = 2\pi\chi(R).$$

Let us mention some special cases.

**Corollary 13.7** (Special cases of the Gauss–Bonnet Theorem)**.**

(a) *(R bounded by geodesics)* If the region $R$ is bounded piecewise by *geodesics*, then

$$\int_R K \, \mathrm{d}A + \sum_{j=1}^r \vartheta_j = 2\pi\chi(R).$$

(b) *(R bounded by a closed geodesic)* If $\gamma$ is a simple closed geodesic and $R$ is a region having $\gamma$ as its boundary, then

$$\int_R K \, \mathrm{d}A = 2\pi\chi(R).$$

(c) *(No boundary, case $R = S$, $\partial R = \emptyset$)* If $S$ is a closed surface, then

$$\int_S K \, \mathrm{d}A = 2\pi\chi(S).$$

**Theorem 13.8** (Local Gauss–Bonnet Theorem/Gauss–Bonnet Theorem for triangles)**.** Let $T$ be a triangle in an oriented surface $S$ with interior angles $\alpha$, $\beta$ and $\gamma$. Then

$$\int_T K \, \mathrm{d}A + \int_{\partial T} \kappa_g \, \mathrm{d}s = \alpha + \beta + \gamma - \pi.$$

Some more special cases.

**Corollary 13.9.** Assume that $S$ is a surface of constant Gauss curvature $K$. Assume additionally, that $T$ is a geodesic triangle in $S$ (i.e., $\partial T$ consists of three arcs of geodesics). Then

$$K \cdot (\operatorname{area} T) = \alpha + \beta + \gamma - \pi.$$

**Example 13.10.**

(a) On a sphere ($K = 1$), the sum of angles in a (geodesic) triangle is always *larger* than $\pi$ and the difference is equal to the area of the triangle.

(b) On a plane ($K = 0$), the sum of angles in a (geodesic) triangle is always $\pi$ (independent of the area of the triangle).

(c) On the hyperbolic plane ($K = -1$), the sum of angles in a (geodesic) triangle is always *smaller* than $\pi$ and the difference is equal to the area of the triangle.

**Example 13.11.** (a) The total Gauss curvature of the region $R$ of a unit sphere given by the triangle with vertices at the North pole and two points on the equator at distance one quarter of the circumference is equal to $\pi/2$ as $R$ covers one eighth of the surface of the unit sphere. On the other hand, one can observe that $R$ is a regular right-angled triangle, so the statement of the local Gauss–Bonnet theorem becomes "area of $R = 3\pi/2 - \pi$".

(b) The total Gauss curvature of a surface $T$ homeomorphic to a torus is equal to zero since the Euler characteristic is zero. In particular, if $T$ is not flat everywhere, then it contains elliptic, parabolic and flat points.

**Example 13.12.** Let $S$ be homeomorphic to the plane $\mathbb{R}^2$, and assume that $K \leq 0$ everywhere on $S$. Then $S$ cannot have any simple closed geodesic.

Indeed, by the Jordan curve theorem, a simple closed curve $\boldsymbol{\alpha}$ encloses two regions, one of them homeomorphic to a disc; call this region $R$. If we assume now that $\boldsymbol{\alpha}$ were a closed geodesic, then its geodesic curvature would vanish and there would be no vertices, hence by the Gauss–Bonnet theorem we would have

$$\int_R K\, \mathrm{d}A + \underbrace{\int_{\partial R} \kappa_{\mathrm{g}}\, \mathrm{d}s}_{=0} + \underbrace{\sum_{j=1}^r \vartheta_j}_{=0} = 2\pi \underbrace{\chi(R)}_{=1}$$

as the Euler characteristic of a disc is $\chi(R) = 1$ (the same as for a triangle). But since $K \leq 0$, the integral $\int_R K\, \mathrm{d}A \leq 0$, and this is a contradiction. Therefore, there is no such geodesic.

**Example 13.13.** One can verify the local Gauss–Bonnet theorem explicitly for an "ideal" triangle on a hyperbolic plane: the area of the region bounded by two vertical lines $u = u_1$ and $u = u_2$ and a semicircle intersecting the real axis at points $u_1$ and $u_2$ is equal to $\pi$.

**Example 13.14.** One can verify global Gauss–Bonnet theorem explicitely on a slice of a unit sphere $x^2 + y^2 + z^2 = 1$ between the equator and the parallel $z = 1/2$. Notice that one needs to pay attention to the orientation of each of the boundary componentswhen computing integral of geodesic curvature.

# 14  Hilbert's theorem

(Non-examinable, but useful for recap matherial, see also Do Carmo 5-11)

**Definition 14.1.** A surface $S$ is *immersed* in $\mathbb{R}^3$ if it is smoothly embedded, posibly with self-intersections, i.e. if for every point $p \in S$ there exists a local parametrisation $x : U \to S$ where $x$ is a local diffeomorphism and $U \in \mathbb{R}^2$ is an open set with $x \in U$.

**Theorem 14.2** (Hilbert's Theorem)**.** There is no isometric immersion of the complete hyperbolic plane to $\mathbb{R}^3$.

In particular, there is no isometric embeddings (i.e. without self-intersections) of the complete hyperbolic plane to $\mathbb{R}^3$.

**Remark 14.3.** "Complete" here means "complete as a metric space" (i.e. every Cauchy sequence converges inside the space).

**Remark 14.4.** Recall the upper half-plane model of the hyperbolic plane: the upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the first fundamental form given by $E = G = 1/y^2$, $F = 0$.