Geometry III/IV

Circles in Euclidean geometry
Some facts and proofs

Facts ...

Fact 1. For any three non-collinear points on the Euclidean plane $\mathbb{E}^2$ there exists a unique circle passing through these three points.

Fact 1'. Every triangle has a unique circumscribed circle.

Fact 2. Three perpendicular bisectors of a triangle do intersect in one point.

Fact 3. For a circle c and two points X, Y on c there exists a unique line or circle orthogonal to c and passing through X and Y.

Fact 4. An inversion with respect to a circle c preserves lines and circles orthogonal to c.

Fact 5. Lines and circles not orthogonal to a circle c are not preserved by inversion with respect to c.

Fact 6. Let c be a circle and let A, B be two points not in c. Then there exists a unique line or circle orthogonal to c and passing through A and B.

Fact 7. Let c be a circle and c' be a line or circle $c \perp c'$. Let A be any point not in $c \cap c'$. Then there exists a unique line or circle c_1 passing through A and orthogonal to both c and c'.

Fact 8. Stereographic projection coincides with the restriction to the sphere of a suitable inversion.

Fact 9. Let AB be a diameter of a circle c and C any other point. Then $\angle ACB = \pi/2$ if and only if $C \in c$.

... and Proofs

Fact 1. For any three non-collinear points on the Euclidean plane $\mathbb{E}^2$ there exists a unique circle passing through these three points.

Proof. Let A, B, C be three non-collinear points. Let M be a midpoint of the segment AB and let N be a midpoint of BC. Let l_1 be a line through M orthogonal to AB and let l_2 be a line through N orthogonal to BC. Since A, B and C are not collinear, l_1 is not parallel to l_2. Denote by O their intersection point $O = l_1 \cap l_2$ (see Fig.1.a).

The triangles OMA and OMB are congruent (by SAS: $OM = OM$, $\angle OMA = \angle OMB = \pi/2$, $MA = MB$). This implies $OA = OB$. Similarly, using triangles ONB and ONC we get $OB = BC$. Hence, the circle of radius OA centered at O passes through all three points A, B and C.

We are left to show that a circle through three given points is unique. It is clear that there is no other circle through the same three points centered at the same point O.

Suppose there is a circle centered at another point O'. Let M' and N' be orthogonal projections of O' to AB and BC respectively (i.e. $O'M' \perp AC$ and $O'N' \perp BC$), see Fig. 1.b). Then right-angled triangles $O'M'A$ and $O'M'C$ are congruent since $AO' = CO'$ and $O'M' = O'M'$. This implies $AM' = M'C$. Hence, $O'M' \in l_1$ as $O'M' \perp AC$ and M' is a midpoint of AC. By the similar reasons, $O'N' \in l_2$. So $O' = l_1 \cap l_2$. However, the line l_1 and l_2 have a unique intersection points O. This contradicts to the assumption that there exists a circle through A, B, C centered at $O' \neq O$. Hence, the circle through three non-collinear three points is unique.

□

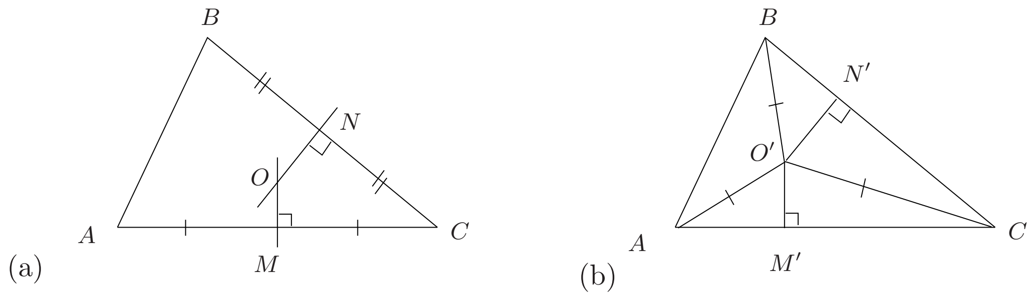

FIGURE 1

We can reformulate Fact 1 as follows:

Fact 1'. Every triangle has a unique circumscribed circle.

Remark. The lines l_1 and l_2 in the proof of Fact 1 are called *perpendicular bisectors*. (A *perpendicular bisector* for a segment is a line orthogonal to the segment and passing through its midpoint).

Fact 2. Three perpendicular bisectors of a triangle do intersect in one point.

Proof. Let l_1, l_2, l_3 be the three perpendicular bisectors. Then each two of them intersect at the center of the circumscribed circle, so all three of them pass through the center of the circumscribed circle.

□

Remark. A very similar reasoning involving angle bisectors allows to prove the following to statements:

- Every triangle has an inscribed circle.
- Three angle bisectors of a triangle do intersect in one point.

(Here an inscribed circle is a circle tangent to all three sides of a triangle and an angle bisectors is a ray decomposing the angle into two equal angles).

Fact 3. For a circle c and two points X, Y on c there exists a unique line or circle orthogonal to c and passing through X and Y.

Proof. Denote by O the center of the circle and by l_X and l_Y the tangent lines at point X and Y.

First, suppose that the points X and Y do not lie on one diameter. Then l_X is not parallel to l_Y. Denote $O' = l_X \cap l_Y$. Consider a circle c' of radius $O'X$ centered at O'. It is orthogonal to l_X and l_Y as $O'X$ and $O'Y$ are its radii. Hence c' is a circle orthogonal to c and passing through X and Y.

To show uniqueness of such a circle, suppose there is another one centered at $O'' \neq O'$. The $O''X \perp OX$ and $O''Y \perp OY$ which implies that $O'' \in l_X \cap l_Y$. However, the unique intersection point of l_X and l_Y is O'. The contradiction implies uniqueness.

Suppose now that X and Y do lie on one diameter. Then l_1 does not intersect l_2, so there is no circle through X and Y orthogonal to c. So, the line XY is a unique line or circle through X and Y orthogonal to c.

□

Remark. (Another proof of the uniqueness.)
Suppose that there are two lines or circles c_1, c_2 orthogonal to c and passing through X and Y. Consider a Möbius map f taking X and Y to 0 and ∞. f maps c_1, c_2 and c to three (distinct) lines through 0, two of them orthogonal to the third. This is impossible.

Fact 4. An inversion with respect to a circle c preserves lines and circles orthogonal to c.

Proof. Let c' be a circle or line orthogonal to c and let X and Y be two intersection points of c and c' (there are two distinct points, otherwise c' is tangent to c, but not orthogonal). Inversions preserves angles. So, inversion with respect to c takes c' to a circle or line orthogonal to c and passing through X and Y. In view of uniqueness of such a line, we get $f(c') = c'$, as required.

□

Fact 5. Lines and circles not orthogonal to a circle c are not preserved by inversion with respect to c.

Proof. Let I_c be inversion with respect to c. Let c' be a line or circle not orthogonal to c. If c' does not intersect c or intersect c in a unique point (i.e. c' is tangent to c) then c' lies either entirely inside the disk bounded by c or entirely onside the disk. This means that I_c takes c' outside (respectively inside) the disk and does not preserve it.

Suppose now that c' intersects c in two points, c' is not orthogonal to c and $I_c(c') = c'$. Then I_c takes an acute angle formed by c and c' to a obtuse one, which contradicts to the fact that inversion preserves angles. The contradiction shows that $I_c(c') \neq c'$.

□

Fact 6. Let c be a circle and let A, B be two points not in c. Then there exists a unique line or circle orthogonal to c and passing through A and B.

Proof. Let I_c be an inversion with respect to c. Denote $A' = I_c(A)$.

If the points A, A' and B are collinear, let c' be a line containing these three points. Otherwise, by Fact 1 there exists a unique circle c' through A, A' and B. Notice that the line or circle c' intersects c at some point X (one of the points A and A' is inside the disk bounded by c, another is outside). Furthermore, I_c preserves X and swaps A and A', in other words, I_c preserves the triple A, A', X. Hence, $I_c(c') = c'$ (by uniqueness of line or circle through three points). In view of Fact 5 this implies that c' is orthogonal to c.

We are left to prove uniqueness of c'. Suppose that there exists another line or circle c'' orthogonal to c and passing through A and B. Since $c'' \perp c$, we have

$I_c(c'') = c''$ (Fact 4). Hence, $I_c(A) \in c''$, so that c'' is a circle or a line through A, A', B. By uniqueness of such a circle or a line (Fact 1) we conclude $c'' = c$.

□

Exercise. Let c be a circle, let X be a points on c and A be a point not on c. Show that there exists are unique line or circle orthogonal to c and passing through X and A.

Fact 7. Let c be a circle and c' be a line or circle $c \perp c'$. Let A be any point not in $c \cap c'$. Then there exists a unique line or circle c_1 passing through A and orthogonal to both c and c'.

Proof. Suppose that $A \notin c'$ (the case $A \in c'$, $A \notin c$ is similar). Let $A' = I_{c'}(A)$ be the image of A under the inversion with respect to c'. Let c_1 be a circle or line passing through A and A' and orthogonal to c (it does exists by Fact 6). Then $I_{c'}(c_1) = c_1$ (since $I_{c'}$ swaps points A and A' and preserves $c_1 \cap c$). This implies (by Fact 5) that c_1 is orthogonal to c'. As $c_1 \perp c$ and $A \in c_1$, the existance part is proved.

To prove uniqueness, notice that the line or circle orthogonal to c' and containing A should also contain $I_{c'}(A)$. So, we are searching for a line or circle containing A, A' and orthogonal to c. By Fact 6 this line is unique.

□

Fact 8. Let S be a sphere centered at O, let Π be a plane through O. Let $l = \pi \cap S$ be a great circle and let $N = Pol(l)$ be a point polar to l. Denote by σ the stereographic projection from N to Π. Denote by S_1 a sphere centered at N and containing l, and denote by I_1 the inversion with respect to S_1.

Then the restriction of the inversion I_1 to the sphere S coincides with the stereographic projection σ.

Proof. I_c takes the sphere S to the plane Π (since inversion preserves angles and the sphere S_1 is a bisector of the angle formed by S and Π). I_1 also takes each point A to a point on the ray NA. So, for any point $A \in S$ the inversion I_1 takes A to the unique point of the intersection $\Pi \cap NA$. This is the definition of the stereographic projection.

□

Fact 9. Let AB be a diameter of a circle c and C any other point. Then $\angle ACB = \pi/2$ if and only if $C \in c$.

Proof. First, suppose that $C \in c$. Let O be a center of c. Then triangles ACO and BCO are isosceles. Hence, $\angle ACO = \angle CAO$ and $\angle BCO = \angle CBO$. This implies that

$$\angle ACB = \angle ACO + \angle BCO = \angle CAO + \angle CBO == \angle CAB + \angle CBA.$$

Since $\angle ACB + \angle CAB + \angle CBA = \pi$, we conclude $\angle ACB = \pi/2$.

Now, suppose that C' lies inside the disk bounded by the circle c. Without loss of generality we assume that C is intersection of the ray OC' and the circle c'. Since $\angle C'AB < \angle CAB$ and $\angle C'BA < \angle CBA$, we see that $\angle AC'B > \angle ACB = \pi/2$.

Similar reasoning shows that for a point C'' outside the disk bouded by c we have $\angle AC''B < \angle ACB = \pi/2$.

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