Geometry III/IV
Möbius transformations – outline

A <u>Möbius transformation</u> (or a linear-fractional transformation) is a map
$f : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$ of the form $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

- Möbius transformations form a group generated by az, $z + b$ and $1/z$.
- The group of Möbius transformations acts on $\mathbb{C} \cup \{\infty\}$ triply transitively.
- Möbius transformations take lines and circles to lines and circles.
- Möbius transformations preserve angles.
- Möbius transformations preserve the orientation.
- Any Möbius transformation is a composition of even number of inversions and reflections.

Types of Möbius transformations

A Möbius transformation have either one or two fixed points ($p = 1$ or $p = 2$).

- if $p = 1$ the transformation is called <u>parabolic</u> and is conjugate to $z + b$.
- any non-parabolic Möbius transformation f is conjugate to az;
    - if $|a| = 1$ then f is called <u>elliptic</u> (az is a rotation);
    - if $|a| \neq 1$, $a \in \mathbb{R}$ then f is called <u>hyperbolic</u> (az is a dilation/contraction);
    - if $|a| \neq 1$, $a \notin \mathbb{R}$ then f is called <u>loxodromic</u>.

<u>Inversions</u>

An inversion I_C with respect to a circle C or radius r centred at O takes a point A
to the point A' lying on the ray OA and satisfying $|OA| \cdot |OA'| = r^2$.

- $I_C^2 = Id$
- $I_C(C) = C$
- Inversions take lines and circles to lines and circles.
- Inversions preserve angles.
- Inversion is conjugate to a reflection by another inversion.

<u>Cross-ratio</u>

$$[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4}$$

- Cross-ratio is preserved by inversions and Möbius transformations.
- Cross-ratio is real if and only if the points z_1, z_2, z_3, z_4 lie on one line or circle.

Stereographic projection

A stereographic projection is a projection from the North pole of the sphere to the
horizontal plane.

- The stereographic projection transform circles to circles and lines;
- it preserves the angles;
- it preserves the cross-ratio.