Geometry III/IV

Facts of hyperbolic geometry — outline

Points and lines

- For any two points in $\overline{\mathbb{H}}^2 = \mathbb{H}^2 \cup \partial\mathbb{H}^2$ there exists a line through these two points.
- For any point $A \in \overline{\mathbb{H}}^2$ and any line l there exists a unique line orthogonal to l and passing through A.
- For any line l and a point $A \notin l$, $A \in \overline{\mathbb{H}}^2$ there exists infinitely many lines l' through A such that $l \cap l' = \emptyset$.

Isometries

- There exists an isometry of $\mathbb{H}^2$ which takes
  - any three points of the absolute to any other three points of the absolute;
  - any point of $\mathbb{H}^2$ to any other point of $\mathbb{H}^2$.
- An isometry of $\mathbb{H}^2$ preserving three points of $\partial\mathbb{H}^2$ is identity.
- The orientation preserving isometries of $\mathbb{H}^2$ in the Poincaré disk model are linear-fractional maps preserving the disk.
- Any isometry of $\mathbb{H}^2$ in the upper half-plane model has either the form $z \mapsto \frac{az+b}{cz+d}$ or the form $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ where $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$.

Parallel lines

Two lines are <u>parallel</u> if they have a common point in $\partial\mathbb{H}^2$.
<u>Angle of parallelism</u> for a line l and a point $A \notin l$ is a half of the angle between two rays with vertex A parallel to l.

- Let AH be a line perpendicular to l, $(H \in l)$. Then a ray AX intersects l iff $\angle XAH < \varphi$, where φ is the angle of parallelism.
- If $d(A, l) = a$ then the angle of parallelism φ satisfies $\cosh a = \frac{1}{\sin\varphi}$ (here $d(A, l) = \min\limits_{B \in l}(d(A, B)) = d(A, H)$ is the distance from A to l).

Ultraparallel lines

- Two lines which are neither parallel nor intersecting are called <u>ultraparallel</u>.
- Any pair of ultraparallel lines have a unique common perpendicular.

Hyperbolic trigonometry

- $\cosh x = \frac{e^x + e^{-x}}{2}$ $\qquad$ $\sinh x = \frac{e^x - e^{-x}}{2}$ $\qquad$ $\tanh x = \frac{\sinh x}{\cosh x}$
- $\cosh^2 x - \sinh^2 x = 1$

Triangles

Let α, β, γ be angles opposite to the sides a, b, c respectively.

- $\alpha + \beta + \gamma < \pi$.
- Four laws of congruence of hyperbolic triangles: SSS, SAS, ASA, <u>AAA</u>.
- The base angles of isosceles triangles are equal.
- Pythagorean thm: if $\gamma = \pi/2$ then $\cosh c = \cosh a \cosh b$.
- Law of sines: $\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}$.
- Law of cosines: $\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha$.
- Area of a triangle: $S_{ABC} = \pi - (\alpha + \beta + \gamma)$.

Types of isometries

A reflection with respect to a line l is an orientation-reversing isometry preserving l pointwise and swapping the half-planes into which l divides $\mathbb{H}^2$.

- Any isometry is a composition of at most 3 reflections.
- A non-trivial orientation-preserving isometry f is a composition of 2 reflections.
  The fixed lines l_1 and l_2 of these two reflections may be intersecting, parallel or ultraparallel which results in three types of isometries (elliptic, parabolic, hyperbolic).
- The types may be determined by
  - the fixed points or
  - the value $|a+d|$, where $f(z) = \frac{az+b}{cz+d}$, $ad - bc = 1$ is the expression of f in the upper half-plane model.
- Types of orientation-preserving isometries:

type	elliptic	parabolic	hyperbolic
fixed points	1 fix pt O in $\mathbb{H}^2$	1 fixed pt X in $\partial\mathbb{H}^2$	2 fix pt X, Y in $\partial\mathbb{H}^2$
(l_1, l_2)	intersecting $O = l_1 \cap l_2$	parallel $X = l_1 \cap l_2$	ultraparallel X, Y are endpoints of l, where $l \perp l_1$ and $l \perp l_2$
$\|a+d\|$	< 2	2	> 2
orthogonal curves	circles (centred at O)	horocycles (centred at X)	equidistant curves (to the line l)
f is conjugate to	rotation	$z + b$	az

Circles, horocycles and equidistant curves

- A circle centred at $O \in \mathbb{H}^2$ is a set of points on the same distance from O.
  It is orthogonal to all lines through O.

- A horocycle centred at $X \in \partial\mathbb{H}^2$ is a limit of of a circle whose center O tends to $X \in \partial\mathbb{H}^2$. It is orthogonal to all lines through X.
  A distance from a horocycle to its center is infinite, however, the distance between two horocycles h_1 and h_2 centred at the same point is finite (and equal to the distance from each point of h_1 to h_2).

- An equidistant curve for a line l is a set of points on the same distance from l. It is orthogonal to each line l_1 such that $l \perp l_1$.