Geometry III/IV

Distance in the hyperboloid model: proofs

Remark: this is an optional handout for those who are interested in proofs. You will NOT be required to reproduce it in the exam. Also, you will NOT be required to use/reproduce/remember the formulas from this handout!

Recall that given $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ the *pseudo-scalar product* of u and v is

$$(u, v) = u_1 v_1 + u_2 v_2 - u_3 v_3.$$

The *points* of $\mathbb{H}^2$ are represented by vectors v satisfying $(v, v) < -1$, $v_3 > 0$ (or by arbitrary vector $(v, v) < 0$ considered up to proportionality $v \sim v$ for $\lambda \in \mathbb{R} \setminus 0$).

The *point of the absolute* is represented by a vector v satisfying $(v, v) = 0$ (considered up to proportionality).

The *lines* are intersections of the hyperboloid with planes passing through the origin. An equation for a plain is

$$a_1 x_1 + a_2 x_2 - a_3 x_3 = 0.$$

The line given by this equation will be represented by a vector $a = (a_1, a_2, a_3)$. Then a point x lie on a line a if and only if $(x, a) = 0$.

Theorem 1. *For two points u and v in $\mathbb{H}^2$ one has*

$$\cosh^2 d(u, v) = \left|\frac{(u, v)^2}{(u, u)(v, v)}\right|.$$

Proof. By transitivity of isometry group on $\mathbb{H}^2$ we may assume $u = (0, 0, 1)$. Applying a rotation around this point (in 3-dimensional space it is represented by a rotation around the third coordinate axes) we may assume that $v = (v_1, 0, v_3)$, $v_1^2 - v_3^2 = 1$. We find $d(u, v)$ by definition, as a cross-ratio of four lines.

The line (plane in the model) through u and v has equation $x_2 = 0$, i.e. it is represented by the vector $(0, 1, 0)$. This line intersects the absolute in the points $(x, x) = 0$, $x_2 = 0$, i.e. in $x_1^2 - x_3^2 = 0$ which gives two solutions for $x_3 > 0$: $X = (-1, 0, 1)$ and $Y = (1, 0, 1)$ (see Fig. 1 for the projection of the pattern to the plane $x_3 = 1$). To find the distance $d(u, v)$ we need to find a cross-ratio of four lines spanned by u, v, X and Y.

To find the cross-ratio of four lines we need to intersect all four lines by some line l (and the result does not depend on the choice of l!). Choose l the horizontal line through $(0, 0, 1)$ (it is given by equations $x_3 = 1$, $x_2 = 0$). Renormalizing $v = (v_1, 0, v_3)$ so that it belongs to the plane $x_3 = 1$ we get $v' = (\frac{v_1}{v_3}, 0, 1)$. So, using the line $x_3 = 1$,$x_2 = 0$ we get

$$|[u, v, X, Y]| = |[0, \frac{v_1}{v_3}, -1, 1]| = \left|\frac{v_1/v_3 + 1}{v_1/v_3 - 1} : \frac{0+1}{0-1}\right| =$$

$$= \frac{v_1 + v_3}{v_1 - v_3} = \frac{(v_1 + v_3)^2}{v_1^2 - v_3^2} = \frac{(v_1 + v_3)^2}{1} = (v_1 + v_3)^2,$$

so that

$$d(u, v) = \frac{1}{2} ln|[u, v, X, Y]| = \frac{1}{2} ln(v_1 + v_3)^2 = ln(v_1 + v_3).$$

Since $v_3^2 - v_1^2 = 1$, we have $v_3 = \cosh t$, $v_1 = \sinh t$ (for $t = arccoshv_3$), which implies

$$e^t = \cosh t + \sinh t = v_1 + v_3.$$

At the same time, $e^{d(u,v)} = e^{ln(v_1+v_3)} = v_1 + v_3$, which implies $t = d(u,v)$ and

$$\cosh d(u,v) = \cosh t = v_3.$$

On the other hand,

$$\frac{(u,v)^2}{(u,u)(v,v)} = \frac{v_3^2}{(-1)(-1)} = v_3^2.$$

Thus,

$$\cosh^2 d(u,v) = |\frac{(u,v)^2}{(u,u)(v,v)}|.$$

□

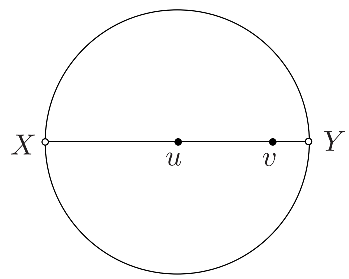

FIGURE 1

The *distance $d(u,l)$ from the point u to the line l* is the minimal distance $d(u,v)$ for between u and a point $v \in l$.

Exercise: show that this minimum do exist.

Lemma 1. *Let u be a point and l be a line represented by a vectors v. Let $t \in l$ be a point such that the line ut is perpendicular to l. Then $d(u,l) = d(u,t)$.*

Proof. Suppose that $d(u,l) = d(u,x)$ for some $x \in l$ and the line ux is not orthogonal to l. consider an orthogonal projection h of x on the line l (i.e. a point $h \in l$ such that xh is orthogonal to l). Then $d(u,x) > d(u,h)$ since ux is a hypotenuse of a right-angle triangle uxh, which contradicts to the choice of x. The contradiction shows that ux is orthogonal to l.

□

Theorem 2. *Let u be a point and l_v be a line represented by a vectors v. Then*

$$\sinh^2 d(u,l_v) = |\frac{(u,v)^2}{(u,u)(v,v)}|.$$

Proof. Let $t \in l_v$ be an orthogonal projection of u to l_v, i.e. the line tu is perpendicular to l_v. By Lemma 1 $d(u,l_v) = d(u,t)$.

Without loss of generality we may assume that $u = (0,0,1)$ and $t = (t_1, 0, t_3)$, $t_1^2 - t_3^2 = -1$ (see Fig. 2 for the projection to the plain $x_3 = 1$). By Theorem 1

$$\cosh^2 d(u,l_v) = cosh^2 d(u,t) = |\frac{t_3^2}{(-1)(-1)}| = t_3^2.$$

Therefore,

$$\sinh^2 d(u,l_v) = \cosh^2 d(u,l_v) - 1 = t_3^2 - 1 = t_1^2.$$

Now, let us find the equation for the line l_v. The line tu corresponds to the plane given by the equation $x_2 = 0$. The whole pattern (i.e. hyperboloid, the point u, the line l_v the line tu) is symmetric with respect to this plane. Hence, the vector v defining the line l_v has zero second coordinate $v_2 = 0$, which implies $v = (v_1, 0, v_3)$. Since the line l_v contains the point $t = (t_1, 0, t_3)$, we have $(v, t) = 0$, i.e. $v_1 t_1 - v_3 t_3 = 0$. This implies $v = \lambda(t_3, 0, t_1)$, or simply $v = (t_3, 0, t_1)$ after rescaling $(v, v) = 1$. Hence,

$$|\frac{(u,v)^2}{(u,u)(v,v)}| = |\frac{t_1^2}{(-1)\cdot 1}| = t_1^2,$$

which coincides with the value of $\sinh^2 d(u, l_v)$.

□

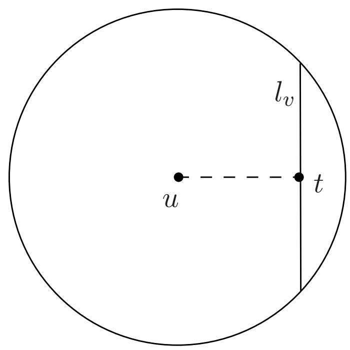

FIGURE 2

By a *distance between two lines* l_1 and l_2 we mean the minimal $d(t_1, t_2)$ for points $t_1 \in l_1$ and $t_2 \in l_2$.

Lemma 2. *Let l_1 and l_2 be two ultra-parallel lines in the hyperbolic plane. Let h be a line orthogonal to both l_1 and l_2. Let $t_1 = h \cap l_1$ and $t_2 = h \cap l_2$ be the intersection point. Then $d(l_1, l_2) = d(t_1, t_2)$.*

Proof. Suppose that $d(l_1, l_2) = d(p, q)$, $p \in l_1$, $q \in l_2$. Suppose that the line pq is not orthogonal to l_1. Let qx be a line orthogonal to l_1, $x \in l_1$. Then $d(q, x) < d(p, q)$ ("the cathetus is shorter than the hypotenuse in the right-angles triangle pqx"). This contradicts to the choice of p and q. Hence, pq is orthogonal to l_1. By the similar reason it is orthogonal to l_2.

□

Theorem 3. *Let l_u and l_v be two ultra-parallel lines represented by vectors u and v respectively. Then*

$$\cosh^2 d(l_u, l_v) = |\frac{(u,v)^2}{(u,u)(v,v)}|.$$

Proof. Let h be a line orthogonal to both l_u and l_v. Let $h_u = h \cap l_u$ and $h_v = h \cap l_v$ be the intersection points. By Lemma 2 $d(l_u, l_v) = d(h_u, h_v)$.

Without loss of generality we may assume $h_u = (0, 0, 1)$ and $h_v = (t_1, 0, t_3)$, $t_1^2 - t_3^2 = 1$ (so that h corresponds to the plane $x_2 = 0$), see Fig. 3 for the projection to the plain $x_3 = 1$. Then l_u and l_v are represented by the vectors $u = (1, 0, 0)$ and $v = (t_3, 0, t_1)$ (since $(h_v, v) = 0$ and $v_2 = 0$). This implies that

$$\cosh^2 d(h_u, h_v) = |\frac{(h_u,h_v)^2}{(h_u,h_u)(h_v,h_v)}| = \frac{t_3}{|t_1^2 - t_3^2|} = |\frac{(u,v)^2}{(u,u)(v,v)}|,$$

This proves the theorem since $d(l_u, l_v) = d(h_u, h_v)$.

$\square$

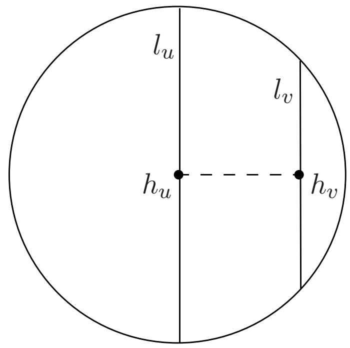

FIGURE 3

Theorem 4. *Let l_u and l_v be two intersecting lines represented by vectors u and v respectively. Let ϕ be an angle formed by these lines. Then*

$$\cos^2\phi = |\frac{(u,v)^2}{(u,u)(v,v)}|.$$

Proof. Applying an isometry, we may assume that the point of intersection of l_u and l_v is $(0,0,1)$. Then the planes through the origin representing the lines l_u and l_v are vertical planes (passing through the third coordinate axis), these planes are represented by vectors $(u_1, u_2, 0)$, $(v_1, v_2, 0)$ (to see that notice, that the vertical planes are symmetric with respect to the plane $x_3 = 0$). Furthermore, due to the rotational symmetry, the angles at the point $(0,0,1)$ are Euclidean angles, i.e. ϕ (or $\pi - \phi$) coincides with the angle between $(u_1, u_2, 0)$ and $(v_1, v_2, 0)$. By Euclidean formula for computation of angles we get

$$\cos\phi = \pm\frac{(u,v)}{\sqrt{(u,u)(v,v)}}$$

(we may use pseudo-scalar product $(\cdot,\cdot)$ in a Euclidean formula since the third coordinate is zero).

$\square$

Theorem 5. *Let l_u and l_v be two distinct lines represented by vectors u and v respectively, $u \neq v$. Then the equality $\frac{(u,v)^2}{(u,u)(v,v)} = 1$ holds if and only if the lines are parallel.*

Proof. Denote $Q := \frac{(u,v)^2}{(u,u)(v,v)}$. It follows from Theorem 4 that in the case of intersecting lines $Q < 1$. Similarly, in by theorem 3 $Q > 1$ in the case of ultra-parallel lines. So, $Q = 1$ implies that the lines are parallel.

To prove the other side notice, that a pair of parallel lines may be obtained as a limit of a pair of intersecting lines, which implies $Q \leq 1$. On the other hand the same pair of parallel lines may be obtained as a limit of a pair of ultra-parallel lines, which implies $Q \geq 1$. Hence, if l_u is parallel to l_v then $Q = 1$.

$\square$