

Groups in Geometries

Geometry	Group G	Generators of G	G preserves...	Transitivity*	Uniqueness**	Classification***	Fixpoints
\mathbb{E}^2	$\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ $A \in O(2, \mathbb{R})$	reflections	distance angles	on flags	3 non-collinear pts	reflection rotation translation glide reflection	line 1 point - -
S^2	$O(3, \mathbb{R})$	reflections	distance angles	on flags	3 non-collinear pts	reflection rotation glide reflection	line 2 (antipodal) points -
Aff	$\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ $A \in GL(2, \mathbb{R})$	$Isom(\mathbb{E}^2)$ and $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \alpha \in \mathbb{R} \setminus \{0\}$	collinearity \Rightarrow parallelism ratios of lengths on a line concurrence of lines ratios of areas	on triangles	3 non-collinear pts		
\mathbb{RP}^1	$PGL(2, \mathbb{R})$	projections of lines to lines	cross-ratio	on triples of points	3 points	$\frac{ax+b}{cx+d}, \quad a, b, c, d \in \mathbb{R}$ $ad - bc \neq 0$	
\mathbb{RP}^2	$PGL(3, \mathbb{R})$	projections of planes to planes	cross-ratio of 4 collinear points	on quadrilaterals (4pts, no 3 collinear)	4 points (no 3 on a line)		
$Möb$	$PGL(2, \mathbb{C})$	$az, z + 1, 1/z$ $(a \in \mathbb{C})$	cross-ratio angles	on triples of pts	3 points	parabolic, conj. to $z + 1$ non-parabolic, conj. to az elliptic $ a = 1$ hyperbolic $ a \neq 0, a \in \mathbb{R}$ loxodromic $ a \neq 1, a \notin \mathbb{R}$	1 point 2 points no attractors/repellers attractor & repeller attractor & repeller
\mathbb{H}^2	$G^+ = PGL(2, \mathbb{R})$	reflections	distance angles	on flags on ideal triangles	3 non-collinear pts 3 pts on absolute	reflection rotation parabolic translation hyperbolic translation glide reflection	line 1 point 1 point on absolute 2 points on absolute 2 points on absolute

*Transitivity = “ G acts transitively on ...”

**Uniqueness = “ $g \in G$ is uniquely determined by the images of ...”

***Classification = “types of elements of G ”

$$PGL(n, k) = GL(n, k) / \pm I$$

$n = \text{dimension}, k = \mathbb{R}, \mathbb{C}$

G^+ = or.preserving subgroup of G