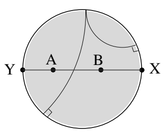
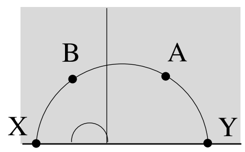
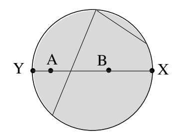
Dr. Anna Felikson, Durham University **Geometry**, 19.2.2015

Models of hyperbolic geometry

model	Poincaré disk	Upper half-plane	Klein disk	two-sheet hyperboloid
$\mathbb{H}^2$	$\{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$	$\{z \in \mathbb{C} \mid Imz > 0\}$	$\{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$	$\{v \in \mathbb{R}^{2,1} \mid (v,v) = -1,\ v_3 > 0\}$ where $(v,u) = v_1u_1 + v_2u_2 - v_3u_3$
$\partial\mathbb{H}^2$ (absolute)	$\{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$	$\{z \in \mathbb{C} \mid Imz = 0\}$	$\{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$	$\{v \in \mathbb{R}^{2,1} \mid (v,v) = 0,\ v_3 > 0\}$ $v \sim \lambda v$
lines				$\{v \mid (v,a) = 0\}$ where $(a,a) > 0$
distance	$d(A,B) = \lvert ln[A,B,X,Y]\rvert$ $X, Y =$ the "endpoints" of the line AB		$d(A,B) = \frac{1}{2}\lvert ln[A,B,X,Y]\rvert$	$d(A,B) = \frac{1}{2}\lvert ln[A,B,X,Y]\rvert$ cross-ratio of four lines*
formula		$\cosh d(u,v) = 1 + \frac{\lvert u-v\rvert^2}{2Im(u)Im(v)}$		$Q = \lvert\frac{(u,v)^2}{(u,u)(v,v)}\rvert$ if $(u,u) < 0$, $(v,v) < 0$ $Q = \cosh^2 d(pt,pt)$ if $(u,u) < 0$, $(v,v) > 0$ $Q = \sinh^2 d(pt,line)$ if $(u,u) > 0$, $(v,v) > 0$ $Q < 1$, intersecting lines $Q = \cos^2\alpha$ $Q = 1$, parallel lines $Q > 1$, ultraparallel lines $Q = \cosh^2 d(line,line)$
isometries**	Möbius transformations		Projective tr	Linear transformations of $\mathbb{R}^{2,1}$
orientation-preserving isometries		$\frac{az+b}{cz+d}$ $a,b,c,d \in \mathbb{R},\ ad - bc = 1$		
orientation-reversing isometries		$\frac{a\bar{z}+b}{c\bar{z}+d}$ $a,b,c,d \in \mathbb{R},\ ad - bc = -1$		
reflections	Euclidean inversions or reflections			$r_a(v) = v - 2\frac{(v,a)}{(a,a)}a$
circles	Euclidean circles		ellipses	plane sections of the hyperboloid
angles	angles=Euclidean angles		distorted angles good for right angles***	

* Cross-ratio of four lines lying in one plane and passing through one point is the cross-ratio of four points at which these lines are intersected by an arbitrary line l (it does not depend on l!).

**We only list the type of the transformations not specifying that they preserve the model.

*** See the backside.

***Right angles in the Klein model**.
Let l be a hyperbolic line.
Let $\bar{l}$ be a Euclidean line containing the segment which represents l in the Klein model.
Let $X_1(l)$ and $X_2(l)$ be the endpoints of l (intersections of $\bar{l}$ with the unit circle).
Let $t_1(l)$ and $t_2(l)$ be tangent lines to the unit circle at the points $X_1(l)$ and $X_2(l)$.
Let $T(l) = t_1(l) \cap t_2(l)$ (if $t_1 || t_2$, i.e. l is represented by a diameter, then $T(l)$ is a point at infinity).

Thm. l' is orthogonal to l if and only if $T(l) \in l'$.

In particular, if l is represented by a **diameter**, then $l' \perp l$ if and only if $\bar{l'} \perp \bar{l}$ (in Euclidean sinse).