On Inversions and Möbius transformations

A. Inversion is **NOT** a Möbius transformation.
In particular, inversion changes the orientation and Möbius transformations preserve it.

B. Inversions are related to Möbius transformations in the same way as reflections in $\mathbb{E}^2$ are related to the group of orientation-preserving isometries of $\mathbb{E}^2$ (they preserve almost all the same properties as the elements of *Möb*).

C. Every inversion I may be written as $I = g \circ \bar{z} \circ g^{-1}$, where $g \in$ *Möb*.

> **Proof:** Let g be a Möbius transformation which takes the real line $\mathbb{R}$ to the fixed points Fix_I of I (it does exist!). Then $r := g^{-1} \circ I \circ g$ takes $\mathbb{R}$ to itself pointwise. In particular, $r(\infty) = \infty$, which implies that r takes lines to lines. Also, r preserves angles (as a composition of inversion and Möbius transformations). Furthermore, r swaps the upper and lower half-planes. This implies that r is a reflection with respect to the real line, i.e. $r = g^{-1} \circ I \circ g = \bar{z}$. Hence, $I = g \circ \bar{z} \circ g^{-1}$.

D. As inversions (and reflections) preserve absolute values of cross-ratios, it is natural to consider an *extended Möbius group* containing both inversions and Möbius transformations: one can generate this group by $< \bar{z}, z + 1, az, 1/z >$ or by $< 1/\bar{z}, z + 1, az, 1/z >$.
As inversions are conjugate to reflections by Möbius transformations, these two ways to generate the group are equivalent. (There is no standard terminology for this group).

E. Every composition of Möbius transformations, reflections and inversions can be written either as $\frac{az+b}{cz+d}$ or as $\frac{a\bar{z}+b}{c\bar{z}+d}$, where $ad - bc \neq 0$.

> **Proof:** Every inversion and reflection is conjugate by a Möbius transformation to $\bar{z}$. Hence, a composition of Möbius transformations, reflections and inversions can be written as a composition of Möbius transformations and complex conjugations. If there is even number of complex conjugations in the composition, we obtain $\frac{az+b}{cz+d}$, otherwise, we obtain $\frac{a\bar{z}+b}{c\bar{z}+d}$.

F. Every isometry of the Poincaré disc can be written as either $\frac{az+b}{cz+d}$ (Möbius transformation) or $\frac{a\bar{z}+b}{c\bar{z}+d}$ (anti-Möbius transformation), where $Ad - bc \neq 0$..

> **Proof:** Let $f \in Isom(\mathbb{H}^2)$ be an isometry. Consider a flag F. By Theorem 6.12 an isometry of the Poincaré disc is uniquely determined by an image of a flag. We can construct a map taking the flag F to $f(F)$ as a composition of Möbius transformations with inversions and reflections: see the proof of Theorem 6.7(1) for the construction of the map taking the given point to the centre of the disc, then apply rotations about the centre, and reflections with respect to the line through the centre if needed. Statement E implies that such a transformation is either Möbius or anti-Möbius.