Dr. Anna Felikson, Durham University **Geometry**, 22.1.2017

Questions from Problems classes

Problems Class 1 (24 October 2016)

Using reflections for solving problems. Geometric constructions.

1. Let $R_{A,\varphi}, R_{B,\psi}$ be rotations about points $A, B \in E^2$ by angles $0 \leq \varphi, \psi \leq \pi$. What is $R_{B,\psi} \circ R_{A,\varphi}$?
2. Let l be a line and A, B be two points not on l. Find a shortest path from A to B passing through at least one point of l.
3. Given ruler and compass, construct:
   (a) a perpendicular bisector to a given segment AB;
   (b) a line passing through a given point A and perpendicular to a given line l;
   (c) a midpoint for a given segment AB;
   (d) an angle bisector;
   (e) a circumscribed circle for a given triangle;
   (f) an inscribed circle for a given triangle.

   Remark. Not everything is constructible!
   In particular, squaring a circle, duplicating a cube and trisecting an angle are not.
   (See G. Jones, Algebra and Geometry, Lecture notes, Section 8).

Problems Class 2 (7 November 2016)

Discrete group actions

0. Let $G = \langle g_1, \ldots, g_n \rangle$ be the group generated by $g_1, \ldots g_n \in Isom(\mathbb{E}^n)$. Then $G : \mathbb{E}^2$.
1. Let $G = \langle r_1, r_2 \rangle$ be the group generated by two reflections on $\mathbb{E}^2$. When G is discrete?
2. Let G be a group generated by reflection with respect to the sides of isosceles right-angled triangle.
   (a) Show that $G : \mathbb{E}^2$ is discrete.
   (b) Find a fundamental domain for the action;
   (c) Find the orbit-space for the action (see Def. 1.28 in the outlines).
4. Find the orbit-spaces for the following actions:
   (a) $\mathbb{Z} : \mathbb{E}^1$ be shifts, $f_n(x) = x + n$;
   (b) $\mathbb{Z}^2 : \mathbb{E}^2$ be translations in two non-collinear directions.
5. $G = \langle r_1, r_2 \rangle$, where r_1, r_2 are reflections with respect to two lines forming an angle $2\pi/3$. Find the fundamental domain for the action $G : \mathbb{E}^2$.

Problems Class 3 (21 November 2016)
Spherical geometry

1. Show that the fixed points of isometries on S^2 distinguish the types of isometries.
2. Let G be a reflection, $h \in Isom(S^2)$. Show that g is conjugate to h in $Isom(S^2)$ if and only if h is a reflection.
3. Let T be a regular tetrahedron.
   (a) Find the group $G = Isom(T)$ of symmetries of T.
   (b) Show that G acts discretely on the unit sphere centred at the centre of T.

   Remark-Definition By a *regular* polyhedron we mean a polyhedron P whose group of symmetries acts transitively on its flags (i.e. on triples (v, e_v, f_e) where v is a vertex of P, e is an edge containing v, and f is a face containing e).
4. (Area of a slice on a sphere).
   Let α and β be two parallel planes crossing the sphere. Let $h = d(\alpha, \beta)$ be the distance between α and β in $\mathbb{E}^3$. Let $A_{\alpha,\beta}$ be the area of the part of the sphere bounded by these planes. Show that $A_{\alpha,\beta}$ only depends on h (and the radius R of S^2) but not on the place where the planes cut the sphere.

Problems Class 3 (5 December 2016)
Projective geometry

1. Find a projective transformation f which takes
   $A = (1:0:0)$ to $(0:0:1)$, $B = (0:1:0)$ to $(0:1:1)$
   $C = (0:0:1)$ to $(1:0:1)$, $D = (1:1:1)$ to $(1:1:1)$
2. In the above notations, let $X = AD \cap BC$. Find $f(X)$.
3. Calculate $[A, B, C, D]$. (It is not defined as they are not collinear!)

3'. Calculate $[A, B, E, F]$, where $E = (1:1:0)$, $F = (1:2:0)$.

4. Check that f preserves $[A, B, E, F]$ (i.e. that $[A, B, E, F] = [f(A), f(B), f(E), f(F)]$
5. Let $A_1, A_2, A_3, A_4 \in a$ be the points lying on the line a, let $B_1, B_2, B_3, B_4 \in b$ be the points lying on the line b. Let $p_i = A_iB_i$. Suppose that the lines p_1, p_2, p_3, p_4 are concurrent. Show that then the points $A_{i+1}B_i \cap A_iB_{i+1}$ are collinear.
6. Write (and prove) the statement dual to one in Question 3.

Problems Class 5 (22 January 2017)
Inversions and Möbius transformations

1. Find the type of the Möbius transformation $f(z) = \frac{1}{z}$.

2. Let f, g be reflections or inversions. Show that $g \circ f = f \circ g$ if and only if Fix_f is orthogonal to Fix_g.

3. I_1 is an inversion with respect to the circle with centre 0 and radius 1. $I_{\sqrt{2}}$ is an inversion with respect to the circle with centre $-i$ and radius $\sqrt{2}$. Show that $r = I_{\sqrt{2}} I_1 I_{\sqrt{2}}$ is a reflection.

4. Let $\gamma_1, \dots, \gamma_5$ be circle all passing through the same points $A, B \in \mathbb{R}^2$. Show that there exists a circle $\mathcal{C}$ orthogonal to all circles γ_i.

5. Prove Ptolemy's Theorem: for a cyclic quadrilateral $ABCD$ holds
   $|AB| \cdot |CD| + |BC| \cdot |DA| = |AC| \cdot |BD|$.

   (three proofs: (a) with cross-ratios, as in HW; (b) with inversion; (c) "proof without words" from http://www.cut-the-knot.org/proofs/PtolemyTheoremPWW.shtml).

Problems Class 6 (6 February 2017)
Poincaré disc model of hyperbolic geometry

0. A hyperbolic isometry is uniquely determined by images of three points on the absolute.

Definition. Two lines on the hyperbolic plane may
- either intersect,
- or have a unique common point (then they are called *parallel*),
- or have no common points in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ (then they are called *divergent* or *ultra-parallel.*

1. Show that any two divergent lines have a unique common perpendicular
   (i.e. if $l_1, l_2 \subset \mathbb{H}^2$ are divergent then there exists a unique line l' such that $l' \perp l_1$ and $l' \perp l_2$).

2. A hyperbolic line is "infinitely long".

Definition. A hyperbolic polygon with all vertices on the absolute is called an *ideal* polygon.
Remark: Ideal polygons have zero angles.

3. (a) Show that up to applying an isometry, there exists a unique hyperbolic ideal triangle.
   (b) Show that hyperbolic ideal quadrilaterals modulo isometries form a 1-parameter family.
   (c) How many hyperbolic ideal n-gons are there?

Problems Class 7 (20 Febuary 2017)
Elementary hyperbolic geometry

1. Compute the area of a disc of radius r.
2. Let $\mathbf{O(r)}$ be a hyperbolic orange of radius r with a pulp of radius $9/10 \cdot r$ and a thin peel which is only $1/10 \cdot r$ thick. Show that for large $r \to \infty$ almost all volume of the orange is the peel.
3. Remarks on regular polygons:
   - **Definition:** a polygon (or more generally, a polytope in n-dim) is regular if the group of its symmetries acts transitively on its flags (where a flag is a vertex A_1, with an edge A_1A_2 from this vertex, with a 2-face $A_1A_2A_3$ through that edge, ..., with an $(n-1)$-face $A_1A_2\dots A_{n-1}$).
   - In 2-dimensional case (for all of $S^2, \mathbb{E}^2, \mathbb{H}^2$): A polygon is regular iff all its angles are of the same size and all its sides are of the same length.
   - No of the angle size or sides length conditions is sufficient alone!
4. Triangles with some vertices at the absolute:
   - AAA congruence does work for all triangles (even for non-compact ones).
   - SSS does not work for triangles with infinite sides!
   - All usual formula work when they make sense (i.e. when entries are finite).
5. (a) Show that any hyperbolic triangle have an inscribed circle.
   (b) Show that not every hyperbolic triangle has a circumscribed circle.
6. Constructions with hyperbolic ruler and circle:
   - midpoint of a segment;
   - perpendicular bisector;
   - angle bisector;
   - perpendicular through a given points A to a given line l;
   - centre of a given circle;
   - inscribed circle for a triangle;
   - circumscribed circle for a triangle (when exists)
   - tangent to a circle.

   I don't know whether it is possible to construct:

   - a common perpendicular to two given ultra-parallel lines.

   if the ruler is of finite length.

Problems Class 8 (6 March 2017)
Computations in the Klein model

1. Use the Klein model to prove that in a right-angled triangle with right angle γ holds:
   (a) $\sinh a = \sinh c \sin \alpha$
   (b) $\tanh b = \tanh c \cos \alpha$
   (c) $\cosh c = \cosh a \cosh b$.
2. Use the Klein model to find the radius of the circle inscribed to the ideal triangle.
3. Let l be a hyperbolic line, e be an equidistant curve to that line. Let $M, N \in l$. Choose a point $A \in e$ and construct $B = e \cap AM$, $C = e \cap AN$.
   (a) Show that the area of the triangle $\triangle ABC$ does not depend on the choice of $A \in e$.
   (b) Formulate and show similar statements in Euclidean and spherical geometries.
   (c) In Euclidean geometry, one can also state that the area of the triangle $\triangle AMN$ does not depend on the choice of A. Does similar statement hold in the hyperbolic case?