

Homework 15-16

Starred problems due on Thursday, 2 March

15.1. (*)

- (a) Let P and Q be feet of the altitudes in an ideal hyperbolic triangle. Find PQ .
- (b) Find the radius of a circle inscribed into an ideal hyperbolic triangle.
- (c) Show that a radius of a circle inscribed into a hyperbolic triangle does not exceed $\operatorname{arcosh}(2/\sqrt{3})$.

15.2. For a right hyperbolic triangle ($\gamma = \frac{\pi}{2}$) show:

$$(a) \tanh b = \tanh c \cos \alpha, \quad (b) \sinh a = \sinh c \sin \alpha.$$

15.3. Show that in the upper half-plane model the following distance formula holds:

$$2 \sinh^2 \frac{d}{2} = \frac{|z - w|^2}{2 \operatorname{Im}(z) \operatorname{Im}(w)}$$

15.4. Find an area of a right-angled hyperbolic pentagon.

15.5. (*) In the upper half-plane model, find the locus of points z lying on distance d from the line 0∞ .16.1. In the Klein disc model draw two parallel lines, two ultra-parallel lines, an ideal triangle, a triangle with angles $(0, \frac{\pi}{2}, \frac{\pi}{3})$.

16.2. (*) Show that three altitudes of a hyperbolic triangle either have a common point or are pairwise parallel or there is a unique line orthogonal to all three altitudes.

16.3. Let u, v be two vectors in $\mathbb{R}^{2,1}$. Denote $Q = |\frac{(u,v)^2}{(u,u)(v,v)}|$, where $(x, y) = x_1y_1 + x_2y_2 - x_3y_3$. Show the following distance formulae:

- (a) if $(u, u) < 0$, $(v, v) < 0$, then u and v give two points in \mathbb{H}^2 , and $\cosh^2(u, v) = Q$.
- (b) if $(u, u) < 0$, $(v, v) > 0$, then u gives a point and v give a line l_v on \mathbb{H}^2 , and $\sinh^2 d(u, l_v) = Q$.
- (c) if $(u, u) > 0$, $(v, v) > 0$ then u and v define two lines l_u and l_v on \mathbb{H}^2 and
 - if $Q < 1$, then l_u intersects l_v forming angle φ satisfying $Q = \cos^2 \varphi$;
 - if $Q = 1$, then l_u is parallel to l_v ;
 - if $Q > 1$, then l_u and l_v are ultra-parallel lines satisfying $Q = \cosh^2 d(l_u, l_v)$.

16.4. (*) Consider the two-sheet hyperboloid model $\{u = (u_1, u_2, u_3) \in \mathbb{R}^{2,1} \mid (u, u) = -1, u_3 > 0\}$, where $(u, u) = u_1^2 + u_2^2 - u_3^2$.

(a) For the vectors

$$\begin{array}{lll} v_1 = (2, 1, 2) & v_2 = (0, 1, 2) & v_3 = (3, 4, 5) \\ v_4 = (1, 0, 0) & v_5 = (0, 1, 0) & v_6 = (1, 1, 2) \end{array}$$

decide if v_i corresponds to a point in \mathbb{H}^2 , or a point in the absolute, or a line in \mathbb{H}^2 .

- (b) Find the distance between the two points of \mathbb{H}^2 described in (a).
- (c) Which pair the lines in (a) is intersecting? Which lines are parallel? Which are ultra-parallel?
- (d) Find the distance between the pair of ultra-parallel lines in (a).
- (e) Does any of the points in (a) lie on any of the three lines?
- (f) Find the angle between the pair of intersecting lines.

References:

1. Lectures 29-32 (Elementary hyperbolic geometry, area, Klein model and hyperboloid model) are based on Lectures VII, VIII, VI and XIII of Prasolov's book.