Homework 17-18

Starred problems due on Tuesday, 14 March

17.1. Prove that any pair of parallel lines can be transformed to any other pair of parallel lines by an isometry.

17.2. Let $A, B \in \gamma$ be two points on a horocycle γ. Show that the perpendicular bisector to AB is orthogonal to γ.

17.3. Let f be a composition of three reflections. Show that f is a glide reflection, i.e. a hyperbolic translation along some line composed with a reflection with respect to the same line.

17.4. (*) Given an isometry f of the hyperbolic plane such that the distance from A to $f(A)$ is the same for all points $A \in \mathbb{H}^2$, show that f is an identity map.

17.5. (*) Let a and b be two vectors in the hyperboloid model such that $(a, a) > 0$ and $(b, b) > 0$. Let l_a and l_b be the lines determined by equations $(x, a) = 0$ and $(x, b) = 0$ respectively. And let r_a and r_b be reflections with respect to l_a and l_b.

(a) For $a = (0, 1, 0)$ and $b = (1, 0, 0)$ write down r_a and r_b.
Find $r_b \circ r_a(v)$, where $v = (0, 1, 2)$.

(b) What type is the isometry $\phi = r_b \circ r_a$ for $a = (1, 1, 1)$ and $b = (1, 1, -1)$?
(Hint: you don't need to compute r_a and r_b).

(c) Find an example of a and b such that $\phi = r_b \circ r_a$ is a rotation by $\pi/2$.

18.1 Let l be a line on the hyperbolic plane and let E_l be the equidistant curve for l.

(a) Let C_1 and C_2 be two connected components of the same equidistant curve E_l. Show that that C_1 is also equidistant from C_2, i.e. given a point $A \in C_1$ the distance $d(A, C_2)$ from A to C_2 does not depend on the choice of A.

(b) Let $A \in E_l$ be a point on the equidistant curve, and let $A_l \in l$ be the point of l closest to A. Show that the line AA_l is orthogonal to the equidistant curve.

(c) Let $P, Q \in l$ be two points on l. Let $A \in E_l$ be a point of the equidistant curve such that the segments AP and AQ contain no point of E_l except A. Continue the rays AP and AQ till the next intersection points with E_l, denote the resulting intersection points by B and C. Let T be a curvilinear triangle ABC (with geodesic sides AB and AC, but BC being a segment of the equidistant curve). Assuming that all angles of ABC are acute show that the area of T does not depend on the choice of $A \in E_l$.

(d) With the assumptions of (c), show that the area of the geodesic triangle ABC does not depend on the choice of A.

18.2. (*)

(a) Let l and l' be ultra-parallel lines. Let γ be an equidistant curve for l intersecting l' in two points A and B. Denote by h the common perpendicular to l and l' and let $H = h \cap l'$ be the intersection point. Show that $AH = HB$.

(b) Let l be a line and γ be an equidistant curve for l. For two points A, B on γ, show that the perpendicular bisector of AB is also orthogonal to l.

(c) Let ABC be a triangle in the Poincare disc model. Let γ be a Euclidean circumscribed circle (i.e. a circumscribed circle for ABC considered as a Euclidean triangle). Suppose that γ intersects the absolute at points X and Y. Show that the (hyperbolic) perpendicular bisector to AB is orthogonal to the hyperbolic line XY.

(d) Show that three perpendicular bisectors in a hyperbolic triangle are either concurrent, or parallel, of have a common perpendicular.

References:

1. Lectures 33, 34 (types of isometries in hyperbolic geometry, horocycles and equidistant curves) are based on Lecture IX of Prasolov's book.

2. Lectures 35, 36 (Poincare theorem, modular group) are based on
   • parts of "Hyperbolic geometry" Caroline Series, especially by some parts of Chapter 6, as well as (small pieces of) Chapters 4 and 5;
   • and on Lectures X and XI of Prasolov's book.

3. In Lectures 37 and 38 we will use material from many different sources, including Prasolov's Lecture XII.