

### Homework 3-4

#### Starred problems due on Thursday, 10 November

3.1 Let  $ABC$  be a triangle and let  $f$  be an isometry. Prove that the points  $C$  and  $D$  lie on the same side with respect to the line  $AB$  if and only if the points  $f(C)$  and  $f(D)$  lie on the same side with respect to the line  $f(A)f(B)$ .

3.2 Let  $f, g \in \text{Isom}(\mathbb{E}^2)$ . Show that  $g(x) \in \text{Fix}_{gfg^{-1}} \Leftrightarrow x \in \text{Fix}_f$  for all  $x \in \mathbb{E}^2$ .

3.3 (\*) Show that the map

$$f(\mathbf{x}) = A\mathbf{x}, \quad A \in GL_2(\mathbb{R}),$$

is an isometry if and only if  $A \in O_2(\mathbb{R})$  (i.e.  $A \in GL_2(\mathbb{R})$ ,  $A^T A = I$ ).

3.4 Let  $f : z \mapsto 2z$ ,  $z \in \mathbb{C}$ . Let  $G$  be a group of transformations of  $\mathbb{E}^2$  generated by  $f$ .

- (a) Does  $G$  act discretely on  $\mathbb{C}$ ? Justify your answer.
- (b) Show that  $G$  acts discretely on  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ .
- (c) Find a fundamental domain for the action  $G : \mathbb{C}^*$ .

3.5 (\*) Let  $P$  be a regular hexagon on  $\mathbb{E}^2$ .

- (a) Find a group  $H$  acting on  $\mathbb{E}^2$  discretely and such that  $P$  is a fundamental domain for the action  $H : \mathbb{E}^2$ . (Describe the group in terms of its generators).
- (b) Let  $G$  be a group generated by reflections with respect to the sides of  $P$ . Show that  $G$  is discrete.

The following three subquestions are a bit more involved and are **not mandatory** for submission. You are very welcome to write down and show me your solution/sketches/ideas but also don't worry if you are not really sure how to do that.

- (c) Find a fundamental domain for  $G$ .
- (d) Is  $H$  a subgroup of  $G$ ? If yes, find its index  $[G : H]$ .
- (e) Describe the orbit space of the action  $H : \mathbb{E}^2$ .  
Hint: if you were not too creative in part (a) you would probably get some space we have already seen in the course.

4.1 Let  $G : \mathbb{E}^2$  be a cyclic group generated by a translation  $T$ . Let  $X$  be an orbit space of  $G : \mathbb{E}^2$ .

- (a) Show that  $X$  is an infinitely long cylinder which admits a Euclidean metric (i.e. each point on  $X$  has a neighbourhood isometric to a domain in  $\mathbb{E}^2$ ).
- (b) Find a closed geodesic on  $X$ ;
- (c) Find an open geodesic on  $X$ .

4.2 Let  $X$  be a torus obtained by identification of opposite sides of the Euclidean square.

- (a) Are there closed geodesics on  $X$ ?
- (b) Are there open ones?

4.3 (\*) Given ruler and compass and a circle  $C$  on the plane, construct the centre of the circle. You can use without proofs and further descriptions the construction of perpendicular bisector for a given segment.

4.4 (\*) Does there exist a map of a domain on the sphere onto a domain on the Euclidean plane that takes the segments of spherical lines into segments of Euclidean lines?

**References:**

1. For material of Lecture 5 look at
  - G. Jones, *Algebra and Geometry*, Lecture notes (Section 1).
2. You will find (almost) all material of Lecture 6 in
  - N. Peyerimhoff, *Geometry III/IV*, Lecture notes (Section 1.6).
3. Material of Lecture 7 may be found in Kiselev's *Geometry / Book II. Stereometry*
  - A. P. Kiselev, *Geometry / Book II. Stereometry*.
4. Starting from Lecture 8 we will follow
  - V. V. Prasolov, *Non-Euclidean Geometry*  
(our Lecture 8 is a first half of Prasolov's Lecture 1).