Dr. Anna Felikson, Durham University

Using Euclidean models to study hyperbolic geometry

One of the difficulties (and beauties) in studying hyperbolic geometry is that we derive information about hyperbolic objects from looking at Euclidean objects (in some model).

Here, I try to give one example of how we think about them at the same time.

Proposition (Theorem 6.7(1)).
The group of isometries $Isom(\mathbb{H}^2)$ acts transitively on the points of $\mathbb{H}^2$.

Proof.

Write	Think
We will show that any given point $A \in \mathbb{H}^2$ we can be mapped by a hyperbolic isometry to the centre O of the disc. Let OA be a segment of hyperbolic line	(it is represented by a Euclidean segment OA).
Let M be a hyperbolic midpoint of the hyperbolic segment OA (it exists by Proposition 6.6).	(M is not a midpoint of OA in Euclidean sense).
Let l' be a hyperbolic line passing through M and orthogonal to OA. It exists by Proposition 6.5.	(i.e. l' is a piece of Euclidean circle passing through M and orthogonal to OA and ∂H^2.)
Consider inversion $I_{l'}$ with respect to Euclidean circle representing hyp. line l'.	($I_{l'}$ would play a role of reflection with respect to l in hyperbolic terms).
$I_{l'}$ is an isometry of the Poincaré disc (since it preserves absolute values of cross-ratios and takes the disc to itself).	
Inversion $I_{l'}$ takes the Euclidean line containing OA to itself (since the line OA is orthogonal to the circle of inversion.	(hence $I_{l'}$ takes hyperbolic line OA to itself)
Being an isometry, $I_{l'}$ takes the point A to another point of the line OA lying on the same hyp. distance $d(M,A)$ from M.	
There is a unique point on distance $d(M,A)$ from M except for A, and it coincides with O (as M is a hyp. midpoint of OA).	
Hence, there is a hyperbolic isometry which takes A to O.	represented by Euclidean inversion