Outline

1 Euclidean geometry

1.1 Isometry group of Euclidean plane, $Isom(\mathbb{E}^2)$.

A distance on a space X is a function $d : X \times X \to \mathbb{R}$, $(A, B) \mapsto d(A, B)$ for $A, B \in X$ satisfying

1. $d(A, B) \geq 0$ $\quad (d(A, B) = 0 \Leftrightarrow A = B)$;
2. $d(A, B) = d(B, A)$;
3. $d(A, C) \leq d(A, B) + d(B, C)$ (triangle inequality).

We will use two models of Euclidean plane:
a Cartesian plane: $\{(x, y) \mid x, y \in \mathbb{R}\}$ with the distance $d(A_1, A_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$;
a Gaussian plane: $\{z \mid z \in \mathbb{C}\}$, with the distance $d(u, v) = |u - v|$.

Definition 1.1. A Euclidean isometry is a distance-preserving transformation of $\mathbb{E}^2$,
i.e. a map $f : \mathbb{E}^2 \to \mathbb{E}^2$ satisfying $d(f(A), f(B)) = d(A, B)$.

Thm 1.2. (a) Every isometry of $\mathbb{E}^2$ is a one-to-one map.
(b) A composition of any two isometries is an isometry.
(c) Isometries of $\mathbb{E}^2$ form a group (denoted $Isom(\mathbb{E}^2)$)
with composition as a group operation.

Example 1.3: Translation, rotation, reflection in a line are isometries.

Definition 1.4. Let ABC be a triangle labelled clock-wise. An isometry f is orientation-preserving
if the triangle $f(A)f(B)f(C)$ is also labelled clock-wise.
Otherwise, f is orientation-reversing.

Proposition 1.5. (correctness of Definition 1.12)
Definition 1.4 does not depend on the choice of the triangle ABC.

Example 1.6. Translation and rotation are orientation-preserving,
reflection and glide reflection are orientation-reversing.

Remark 1.7. Composition of two orientation-preserving isometries is orientation-preserving;
composition of an or.-preserving isometry and an or.-reversing one is or.-reversing;
composition of two orientation-reversing isometries is orientation-preserving.

Proposition 1.8. Orientation-preserving isometries form a subgroup (denoted $Isom^+(\mathbb{E}^2)$) of $Isom(\mathbb{E}^2)$.

Theorem 1.9. Let ABC and $A'B'C'$ be two congruent triangles.
Then there exists a unique isometry sending A to A', B to B' and C to C'.

Corollary 1.10. Every isometry of $\mathbb{E}^2$ is a composition of at most 3 reflections.
(In particular, the group $Isom(\mathbb{E}^2)$ is generated by reflections).

Remark: the way to write an isometry as a composition of reflections is not unique.

Example 1.11: rotation and translation as a composition of two reflections.
Glide reflection as a composition of a reflection in some line and
a translation along the same line (a composition of 3 reflections).

Theorem 1.12. (Classification of isometries of $\mathbb{E}^2$) Every non-trivial isometry of $\mathbb{E}^2$ is of one of the following four types: reflection, rotation, translation, glide reflection.

Problems class 1: a. Example of using reflections to study compositions of isometries (write everything as a composition of reflections, make you choice so that some of them cancel).
b. Example of using reflection to find a shortest way from a point A to a rever and then to a point B on the same bank.
c. Ruler and compass constructions: pependicular bisector, perpendicular from a point to a line, midpoint of a segment, angle bisector, inscribed and circumscribed circles for a triangle.

Definition 1.13. Let f $Isom(\mathbb{E}^2)$. Then the set of fixed points of f is $Fix_f = \{x \in \mathbb{E}^2 \mid f(x) = x\}$.

Example 1.14: Fixed points of Id, reflection, rotation, translation and glide reflection are $\mathbb{E}^2$, the line, a point, $\emptyset$, $\emptyset$ respectively.

Remark. Fixed points together with the property of preserving/reversing the orientation uniquely determine the type of the isometry.

Proposition 1.15. Let $f, g \in Isom(\mathbb{E}^2)$. (a) $Fix_{gfg^{-1}} = gFix_f$;
(b) gfg^{-1} is an isometry of the same type as f.

1.2 Isometries and orthogonal transformations

Proposition 1.15a. A linear map $f : \mathbf{x} \to A\mathbf{x}$, $A \in GL(2, \mathbb{R})$ is an isometry if and only if $A \in O(2)$, orthogonal subgroup of $GL(2, \mathbb{R})$ (i.e. iff $A^T A = I$, where A^T is A transposed).

Proposition 1.16. (a) Every isometry f of $\mathbb{E}^2$ may be written as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{t}$.
(b) The linear part A does not depend on the choice of the origin.

Example 1.17. Orthogonal matrices for a reflection (in the vertical axis) and for a rotation.

Proposition 1.18. Let $f(x) = A\mathbf{x} + \mathbf{t}$ be an isometry.
f is orientation-preserving if $detA = 1$ and orientation-reversing if $detA = -1$.

Exercise 1.19. (a) Show that any two reflections are conjugate in $Isom(\mathbb{E}^2)$.
(b) This is not the case for rotations, translations and glide reflections (there are aditional parameters in that cases).

Proposition 1.20. Geodesics on $\mathbb{E}^2$ are straight lines.

1.3 Discrete groups of isometries acting on $\mathbb{E}^2$

Definition 1.21. A group acts on the set X (denoted $G : X$) if
$\forall g \in G\ \exists f_g$, a bijection $X \to X$, s.t. $f_{gh}(x) = (f_g \circ f_h)(x), \forall x \in X, \forall g, h \in G$.

Example 1.22. Action of $\mathbb{Z}$ on $\mathbb{E}^2$ (generated by one translation);
$Isom(\mathbb{E}^2)$ act on points of $\mathbb{E}^2$, lines in $\mathbb{E}^2$, circles in $\mathbb{E}^2$, pentagons in $\mathbb{E}^2$.

Definition 1.23. An action $G : X$ is transitive if $\forall x_1, x_2 \in X\ \exists g \in G : f_g(x_1) = x_2$.

Example. $Isom(E^2)$ acts transitively on points in $\mathbb{E}^2$ and flags in $\mathbb{E}^2$
(a flag is a triple (p, r, H^+) where p is a point, r is a ray starting from p, and H^+ is a choice of a half-plane with respect to the line containing the ray r);
$Isom(E^2)$ does not act transitively on the circles or triangles.

Definition 1.24. Let $G : X$ be an action.
An orbit of $x_0 \in X$ under the action $G : X$ is the set $orb(x_0) := \bigcup\limits_{g \in G} gx_0$.

Example 1.25. orbits of $O(2) : \mathbb{E}^2$ (circles and one point);
orbits of $\mathbb{Z} \times Z : \mathbb{E}^2$ acting by vertical and horizontal translations (shifts of the integer lattice).

Definition 1.26. An action $G : X$ is discrete if none of its orbits possesses accumulation points, i.e. given an orbit $orb(x_0)$, for every $x \in X$ there exists a disc D_x centred at x s.t. the intersection $orb(x_0) \cap D_x$ contains at most finitely many points.

Example. (a) The action $\mathbb{Z} \times Z : \mathbb{E}^2$ is discrete;
(b) the action of $\mathbb{Z} : \mathbb{E}^1$ by multiplication is not discrete.
(c) Given an isosceles right angled triangle, one can generate a group G by reflections in its three sides. Then $G : \mathbb{E}^2$ is a discrete action.

Definition 1.27. An open connected set $F \subset X$ is a fundamental domain for an action $G : X$ if the sets $gF, g \in G$ satisfy the following conditions:

1) $X = \bigcup\limits_{g \in G} \overline{gF}$ (where $\overline{U}$ denotes the closure of U in X);
2) $\forall g \in G$, $g \neq e$, $F \cap gF = \emptyset$;
3) There are only finitely many $g \in G$ s.t. $\overline{F} \cap \overline{gF} \neq \emptyset$.

Definition 1.28.
An orbit-space X/G for the discrete action $G : X$
is a set of orbits with a distance function $d_{X/G} = \min\limits_{\hat{x} \in orb(x),\ \hat{y} \in orb(y)} \{d_x(\hat{x}, \hat{y})\}$.

Example 1.29. $\mathbb{Z} : \mathbb{E}^1$ acts by translations, $\mathbb{E}^1/Z$ is a circle.
$\mathbb{Z}^2 : \mathbb{E}^2$ (generated by two non-collinear translations), $\mathbb{E}^2/\mathbb{Z}^2$ is a torus.

1.4 3-dimensional Euclidean geometry

Model: Cartesian space (x_1, x_2, x_3), $x_i \in \mathbb{R}$, with distance function

$$d(x,y) = (\sum_{i=1}^{3}(x_i - y_i)^2)^{1/2} = \sqrt{(x-y, x-y)}.$$

Properties: 1. For every plane α there exists a point $A \in \alpha$ and a point $B \notin \alpha$;
2. If two distinct planes α and β have a common point A then they intersect by a line containing A.
3. Given two distinct lines l_1 and l_2 having a common point, there exists a unique plane containing both l_1 and l_2.

Proposition 1.30. For every triple of non-collinear points there exists a unique plane through these points.

Definition 1.31. A distance between a point A and a plane α is $d(A, \alpha) := \min_{X \in \alpha}(d(A, X))$.

Proposition 1.32. $AX_0 = d(A, \alpha)$, $X_0 \in \alpha$ iff $AX_0 \perp l$ for every $l \in \alpha$, $X_0 \in l$.

Corollary. A point $X_0 \in \alpha$ closest to $A \notin \alpha$ is unique.

Definition 1.33. (a) The point $X_0 \in \alpha$ s.t. $d(A, \alpha) = AX_0$ is called an orthogonal projection of A to α. Notation: $X_0 = proj_\alpha(A)$.

(b) Let α be a plane, AB be a line, $B \in \alpha$, and $C = proj_\alpha(A)$.
The angle between the line AB and the plane α is $\angle(AB, \alpha) = \angle ABC$,
Equivalently, $\angle(AB, \alpha) = \min_{X \in \alpha}(\angle ABX)$.

Remark. Definition 1.31 (b) and Remark 1.32 imply that if $AC \perp \alpha$ then $AC \perp l$ for all $l \in \alpha$, $C \in l$.

Definition 1.34. The angle $\angle(\alpha, \beta)$ between two intersecting planes $\alpha$and β is the angle between their normals.

Equivalently, if $B \in \beta$, $A = proj_\alpha(B)$, $C = proj_l(A)$ where $l = \alpha \cap \beta$, then $\angle(\alpha, \beta) = \angle BCA$.

Exercise: 1. Check the equivalences.
2. Let γ be a plane through BCA. Check that $\gamma \perp \alpha$, $\gamma \perp \beta$.
3. Let α be a plane, $C \in \alpha$. Let C be a point s.t. $BC \perp \alpha$.
Let β be a plane through C, $\beta \perp \alpha$. Then $B \in \beta$.

Proposition 1.35. Given two intersecting lines b and c in a plane α, $A = b \cap c$, and a line a, $A \in a$, if $a \perp b$ and $a \perp c$ then $a \perp \alpha$ (i.e. $a \perp l$ for every $l \in \alpha$).

Theorem 1.36. (Theorem of three perpendiculars). Let α be a plane, $l \in \alpha$ be a line and $B \notin \alpha$, $A \in \alpha$ and $C \in l$ be three points. If $BA \perp \alpha$ and $AC \perp l$ then $BC \perp l$.

2 Spherical geometry

Geometry of the surface of the sphere.
Model of the sphere S^2 in $\mathbb{R}^3$: (sphere of radius $R = 1$ centred at $O = (0, 0, 0)$)

$$S^2 = \{\ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

2.1 Metric on S^2

Definition 2.1. A great circle on S^2 is the intersection of S^2 with a plane passing though O.

Remark. Given two distinct non-geometrically opposed points $A, B \in S^2$, there exists a unique great circle through A and B.

Definition 2.2. A distance $d(A, B)$ between the points $A, B \in S^2$ is
πR, if A is diametrically opposed to B, and
the length of the shorter arc of the great circle through A and B, otherwise.
Equivalently, $d(A, B) := \angle AOB \cdot R$ (with $R = 1$ for the case of unit sphere).

Theorem 2.4. The distance $d(A, B)$ turns S^2 into a metric space, i.e. the following three properties hold:

M1. $d(A,B) \geq 0 \quad (d(A,B) = 0 \Leftrightarrow A = B)$;
M2. $d(A,B) = d(B,A)$;
M3. $d(A,C) \leq d(A,B) + d(B,C)$ (triangle inequality).

Remark. Need to prove only the triangle inequality, i.e. $\angle AOC \leq \angle AOB + \angle BOC$.

2.2 Geodesics on S^2

Defin. A curve γ in a metric space X is a geodesic if γ is locally the shortest path between its points.
More precisely, $\gamma(t) : (0,1) \to X$ is geodesic
if $\quad \forall t_0 \in (0,1) \;\; \exists \varepsilon : \;\; l(\gamma(t)|_{t_0-\varepsilon}^{t_0+\varepsilon}) = d(\gamma(t_0 - \varepsilon), \gamma(t_0 + \varepsilon))$.

Theorem 2.5. Geodesics on S^2 are great circles.

Definition 2.6. A geodesic $\gamma : (-\infty, \infty) \to X$ (where X is a metric space) is called
closed if $\quad \exists T \in \mathbb{R} : \; \gamma(t) = \gamma(t+T) \;\; \forall t \in (-\infty, \infty)$;
and open, otherwise.

Example. In $\mathbb{E}^2$, all geodesics are open, each segment is a shortest path.
In S^2, all geodesics are closed, one of the two segments of $\gamma \setminus \{A, B\}$ is the shortest path (another one is not shortest if A and B are not antipodal).

From now on, by lines in S^2 we mean great circles.

Proposition 2.7. Every line on S^2 intersects every other line in exactly two antipodal points.

Definition 2.8. By the angle between two lines we mean an angle between the corresponding planes:
if $l_i = \alpha_i \cap S^2$, $i = 1,2$ then $\angle(l_1, l_2) := \angle(\alpha_1, \alpha_2)$.

Equivalently, $\angle(l_1, l_2)$ is the angle between the lines $\hat{l}_1$ and $\hat{l}_2$, $\hat{l}_i \in \mathbb{R}^3$,
where $\hat{l}_i$ is tangent to the great circle l_i at $l_1 \cap l_2$ as to a circle in R^3.

Proposition 2.9. For every line l and a point $A \in l$ in this line
there exists a unique line l' orthogonal to l and passing through A.

Proposition 2.10. For every line l and a point $A \notin l$ in this line, s.t. $d(A,l) \neq \pi/2$
there exists a unique line l' orthogonal to l and passing through A.

Remark. Writing $d(A,l) \neq \pi/2$ we mean the spherical distance on the sphere of radius $R = 1$.

Definition 2.11. A triangle on S^2 is a union of three points and
a triple of the shortest paths between them.

2.3 Polar correspondence

Definition 2.12. A pole to a line $l = S^2 \cap \Pi_l$ is the pair of endpoints of the diameter DD'
orthogonal to Π_l, i.e. $Pol(l) = \{D, D'\}$.
A polar to a pair of antipodal points D, D' is the great circle $l = S^2 \cap \Pi_l$,
s.t. Π_l is orthogonal to DD', i.e. $Pol(D) = Pol(D') = l$.

Property 2.13. If a line l contains a point A then the line $Pol(A)$ contains both points of $Pol(l)$.

Definition 2.14. A triangle $A'B'C'$ is polar to ABC ($A'B'C' = Pol(ABC)$) if
$A' = Pol(BC)$ and $\angle AOA' \leq \pi/2$, and similar conditions hold for B' and C'.

Theorem 2.15. (Bipolar Theorem)
(a) If $A'B'C' = Pol(ABC)$ then $ABC = Pol(A'B'C')$.
(b) If $A'B'C' = Pol(ABC)$ and $\triangle ABC$ has angles α, β, γ and side lengths a, b, c, then $\triangle A'B'C'$ has angles $\pi - a, \pi - b, \pi - c$ and side lengths $\pi - \alpha, \pi - \beta, \pi - \gamma$.

2.4 Congruence of spherical triangles

Theorem 2.16. SAS, ASA, and SSS hold for spherical triangles.

Theorem 2.17. AAA holds for spherical triangles.

2.5 Sine and cosine rules for the sphere

Theorem 2.18. (Sine rule) $\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$.

Remark. If a, b, c are small than $a \approx \sin a$ and the spherical sine rule transforms into Euclidean one.

Corollary. (Thales Theorem) The base angles of the isosceles triangle are equal.

Theorem 2.19. (Cosine rule) $\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$.

Remark. If a, b, c are small than $\cos a \approx 1 - a^2/2$
and the spherical cosine rule transforms into Euclidean one.

Theorem 2.20. (Second cosine rule) $\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c$.

Remark. (a) If a, b, c are small than $\cos a \approx 1$ and the second cosine rule transforms to $\alpha + \beta + \gamma = \pi$.
(b) For a right-angles triangle with $\gamma = \pi/2$,
sine rule: $\sin b = \sin c \cdot \sin \beta$,
cosine rule: $\cos c = \cos a \cos b$ (Pythagorean Theorem).

2.6 Area of a spherical triangle

Theorem 2.21. The area of a spherical triangle with angles α, β, γ equals $(\alpha + \beta + \gamma - \pi)R^2$,
where R is the radius of the sphere.

Corollary 2.22. $\pi < \alpha + \beta + \gamma \leq 3\pi$,
(where the equality holds only if three vertices of the triangle lie on the same line).

Corollary 2.23. $0 < a + b + c \leq 2\pi$,
(where the equality holds only if three vertices of the triangle lie on the same line).

Corollary. There is no isometry from any small domain of S^2 to a domain on $\mathbb{E}^2$

2.7 More about triangles

(1) In a spherical triangle,
(a) medians, (b) altitudes, (c) perpendicular bisectors, (d) angle bisectors are concurrent.

(2) For every spherical triangle there exists a unique circumscribed and a unique inscribed circles.

2.8 Isometries of the sphere

Example 2.25. Rotation, reflection and antipodal map.

Proposition 2.26. Every non-trivial isometry of S^2 preserving two non-antipodal points A, B
is a reflection (with respect to the line AB).

Proposition 2.27. Given points A, B, C, satisfying $AB = AC$, there exists a reflection r such that
$r(A) = A$, $r(B) = C$, $r(C) = B$.

Example 2.28. Glide reflection, $f = r_l \circ R_{A,\varphi} = R_{A,\varphi} \circ r_l$,
where r_l is a reflection with respect to l and $R_{A,\varphi}$ is a rotation about $A = Pol(l)$.

Theorem 2.29.
1. An isometry of S^2 is uniquely determined by the images of 3 non-collinear points.
2. Isometries act transitively on points of S^2 and on flags in S^2.
3. The group $Isom(S^2)$ is generated by reflections.
4. Every isometry of S^2 is a composition of at most 3 reflections.
5. Every orientation-preserving isometry is a rotation.
6. Every orientation-reversing isometry is either a reflection or a glide reflection.

Theorem 2.30. (a) Every two reflections are conjugate in $Isom(S^2)$.
(b) Rotations by the same angle are conjugate in $Isom(S^2)$.

Remark 2.31. Isometries of S^2 may be described by orthogonal matriaces 3×3.
The subgroup of or.-preserving isometries is $SO(3, \mathbb{R}) = \{A \in M_3 | A^T A = I, detA = 1\}$

3 Affine geometry

We consider the same space $\mathbb{R}^2$ as in Euclidean geometry but with larger group acting on it.

3.1 Similarity group

Similarity group, $Sim(\mathbb{R}^2)$ is a group generated by all Euclidean isometries and scalar multiplications:

$(x_1, x_2) \mapsto (kx_1, kx_2)$, $k \in \mathbb{R}$.

Its elements may change size, but preserve the following properties:
angles, proportionality of all segments, parallelism, similarity of triangles.

Remark. A map which may be written as a scalar multiplication in some coordinates in $\mathbb{R}^2$ is called homothety (with positive or negative coefficient depending on the sign of k).

Example 3.1. Using similarity to prove the following statement:
"A midline in a triangle is twice shorter than the corresponding side."

3.2 Affine geometry

Affine transformations are all transformations of the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where $A \in GL(2, \mathbb{R})$.

Proposition 3.2. Affine transformations form a group.

Example 3.3. Affine map may be a similarity but may be not.

Affine transformations do not preserve length, angles, area.

Proposition 3.4. Affine transformations preserve

(1) collinearity of points;
(2) parallelism of lines;
(3) ratios of lengths on any line;
(4) concurrency of lines;
(5) ratio of areas of triangles (so ratios of all areas).

Proposition 3.5. (1) Affine transformations act transitively on triangles in $\mathbb{R}^2$.
(2) An affine transformation is uniquely determined by images of 3 points.

Example 3.6. Using the affine group to prove that the medians of Euclidean triangle are concurrent.

Theorem 3.7. Every bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserving collinearity of points, betweenness and parallelism is an affine map.

Remark. If f is a bijection $\mathbb{R}^2 \to \mathbb{R}^2$ preserving collinearity,
then it preserves parallelism and betweenness.

Theorem 3.7'. (The fundamental theorem of affine geometry)
Every bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserving collinearity of points is an affine map.

Corollary 3.8. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a bijection which takes circles to circles, then f is an affine map.
If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a bijection which takes ellipses to ellipses, then f is an affine map.

4 Projective geometry

4.1 Projective line, $\mathbb{RP}^1$

Points of the projective line are lines though the origin O in $\mathbb{R}^2$.

Group action: $GL(2, \mathbb{R})$ acts on $\mathbb{R}^2$ mapping a line though O to another line through O.
So, acts on $\mathbb{RP}^2$.

Homogeneous coordinates: a line though the O is determined by a pair of numbers (ξ_1, ξ_2), $(\xi_1, \xi_2) \neq (0, 0)$,
where pairs (ξ_1, ξ_2) and $(\lambda\xi_1, \lambda\xi_2)$ determine the same line, so are considered equivalent.
The ratio $(\xi_1 : \xi_2)$ determine the line and is called homogeneous coordinates of the corresponding point in $\mathbb{RP}_1$.

The $GL(2, \mathbb{R})$-action in homogeneous coordinates writes as

$$A : \ (\xi_1 : \xi_2) \mapsto (a\xi_1 + b\xi_2 : c\xi_1 + d\xi_2), \text{ where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and is called a projective transformation.

Remark. Projective transformations are called this way since they are compositions of projections (of one line to another line from a point not lying on the union of that lines).

Lemma 4.1. Let points $A_2.B_2, C_2, D_2$ of a line l_2 correspond to the points A_1, B_1, C_1, D_1 of the line l_1 under the projection from some point $O \notin l_1 \cup l_2$. Then $\frac{C_1A_1}{C_1B_1} / \frac{D_1A_1}{D_1B_1} = \frac{C_2A_2}{C_2B_2} / \frac{D_2A_2}{D_2B_2}$.

Definition 4.2. Let A, B, C, D be four points on a line l, and let a, b, c, d be their coordinates on l. The value $[A, B, C, D] := \frac{c-a}{c-b} / \frac{d-a}{d-b}$ is called the cross-ratio of these points.

Lemma 4.1'. Projections preserve cross-ratios of points.

Definition 4.3. The cross-ratio of four lines lying in one plane and passing through one point is the cross-ratio of the four points at which these lines intersect an arbitrary line l.

Remark. By Lemma 4.1', Definition 4.3 does not depend on the choice of the line l.

Proposition 4.4. Any composition of projections is a liner-fractional map.

Proposition 4.5. A composition of projections preserving 3 points is an identity map.

Lemma 4.6. Given $A, B, C \in l$ and $A', B', C' \in l'$, there exists a composition of projections which takes A, B, C to A', B', C'.

Theorem 4.7. (a) The following two definitions of projective transformations of $\mathbb{RP}^1$ are equivalent:
(1) Projective transformations are compositions of projections;
(2) Projective transformations are linear-fractional transformations.
(b) A projective transformation of a line is determined by images of 3 points.

4.2 Projective plane, $\mathbb{RP}^2$

Model: Points of $\mathbb{RP}^2$ are lines through the origin O in $\mathbb{R}^3$.
Lines of $\mathbb{RP}^2$ are planes through O in $\mathbb{R}^3$.
Group action: $GL(3, \mathbb{R})$ (acts on $\mathbb{R}^3$ mapping a line though O to another line through O).
Homogeneous coordinates: a line though the O is determined by a triple of numbers (ξ_1, ξ_2, ξ_3), where $(\xi_1, \xi_2, \xi_3) \neq (0, 0, 0)$;
triples (ξ_1, ξ_2, ξ_3) and $(\lambda\xi_1, \lambda\xi_2, \lambda\xi_3)$ determine the same line, so are considered equivalent.
Projective transformations in homogeneous coordinates:
$A: (\xi_1 : \xi_2, \xi_3) \mapsto (a_{11}\xi_1 + a_{12}\xi_2 + a_{13} : a_{21}\xi_1 + a_{22}\xi_2 + a_{23}\xi_3 : a_{31}\xi_1 + a_{32}\xi_2 + a_{33}\xi_3)$,
where $A = (a_{ij}) \in GL(3, \mathbb{R})$.

Remark. (1) A unique line passes through any given two points in $\mathbb{RP}^2$.
(2) Any two lines in $\mathbb{RP}^2$ intersect at a unique point.
(3) A plane through the origin in $\mathbb{R}^3$ may be written as $a_1x_1 + a_2x_2 + a_3x_3 = 0$.
This establishes **duality** between points and lines in $\mathbb{RP}^2$
(the point (a_1, a_2, a_3) is dual to the plane $a_1x_1 + a_2x_2 + a_3x_3 = 0$).

Theorem 4.8. Projective transformations of $\mathbb{RP}^2$ preserve cross-ratio of 4 collinear points.

Definition. A triangle in $\mathbb{RP}^2$ is a triple of non-collinear points.

Proposition 4.9. All triangles of $\mathbb{RP}^2$ are equivalent under projective transformations.

Definition. 4.10. A quadrilateral in $\mathbb{RP}^2$ is a set of four points, no three of which are collinear.

Proposition 4.11. For any quadrilateral in $\mathbb{RP}^2$ there exists a unique projective transformation which takes Q to a given quadrilateral Q'.

Proposition 4.12. A bijective map from $\mathbb{RP}^2$ to $\mathbb{RP}^2$ preserving projective lines is a projective map.

Corollary 4.13. A projection of a plane to another plane is a projective map.

Remark 4.14. (Conic sections).
Quadrics, i.e. the curves of second order on $\mathbb{R}^2$ (ellipse, parabola and hyperbola) may be obtained as conic sections (sections of a round cone by a plane).
All of them are equivalent under projective transformations.

Remark 4.15: topology of the projective plane (contains Möbius band, non-orientable, one-sided).

Remark 4.16: Metric on the projective plane: locally isometric to S^2;
not preserved by projective transformations, so, irrelevant for projective geometry.
Geometry of $\mathbb{RP}^2$ with spherical metric (and a group of isometries acting on the space) is called **elliptic geometry** and has the following **properties:**

1. A unique line passes through any two distinct points;
2. Any two lines intersect in a unique point;
3. Given a line l and a point A (not a pole of l),
   there exists a unique line l' s.t. $A \in l$ and $l' \perp l$.

4.3 Polarity on $\mathbb{RP}^2$ (Non-examinable section!)

Consider a trace of a cone $\mathbf{C} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=z^2\}$ on the projective plane $\mathbb{RP}^2$ - a conic.

Definition. Points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ of $\mathbb{RP}^2$ are called polar with respect to $\mathbf{C}$ if $a_1b_1 + a_2b_2 = a_3b_3$.

Example: points of $\mathbf{C}$ are self-polar.

Definition. Given a point $A \in \mathbb{RP}^2$, the set of all points X polar A is the line $a_1x_1 + a_2x_2 - a_3x_3 = 0$, it is called the polar line of A.

How to find the polar line:

Lemma 4.17. A tangent line to $\mathbf{C}$ at a point $B = (b_1, b_2, b_3)$ is $x_1b_1 + x_2b_2 = x_3b_3$.

Proposition 4.18. Let A be a point "outside" $\mathbf{C}$,
let l_B and l_C be tangents to $\mathbf{C}$ at B and C, s.t. $A = l_B \cap l_C$.
Then BC is the line polar to A.

Proposition 4.19. If $A \in \mathbf{C}$ then the tangent l_A at A is the polar line to A.

Proposition 4.20. Let A be a point "inside" the conic $\mathbf{C}$. Let b and c be two lines through A. Let B and C be the points polar to the lines b and c. Then BC is the line polar to A with respect to $\mathbf{C}$.

Remark 4.21. 1. Polarity generalize the notion of orthogonality.
2. More generally, for a conic $\mathbf{C} = \{\mathbf{x} \in R^3 \mid \mathbf{x}^T A\mathbf{x} = 0\}$, where A is a symmetric 3×3 matrix, the point $\mathbf{a}$ is polar to the point $\mathbf{b}$ if $\mathbf{a}^T A\mathbf{b} = 0$.
3. We worked with a diagonal matrix $A = diag\{1, 1, -1\}$.
4. If we take an identity diagonal matrix $A = diag\{1, 1, 1\}$ we get an empty conic $x^2+y^2+z^2 = 0$, which gives exactly the same notion of polarity as we had on S^2.

4.4 Some classical theorems on projective plane

Remark on projective duality:

point $A = (a_1 : a_2 : a_3)$	$\longleftrightarrow$	line l_A : $a_1x_1 + a_2x_2 + a_3x_3 = 0$
$A \in l_B$	$\longleftrightarrow$	$B \in l_A$
line through A, B	$\longleftrightarrow$	point of intersection: $l_A \cap l_B$
3 collinear points	$\longleftrightarrow$	3 concurrent lines
...	$\longleftrightarrow$	...

Proposition 4.22. (On dual correspondence) The interchange of words "point" and "line" in any statement about configuration of points and lines related by incidence does not affect validity of the statement.

Theorem 4.23. (Pappus' theorem). Let a and b be lines, $A_1, A_2, A_3 \in a$, $B_1, B_2, B_3 \in b$.
Let $P_3 = B_1A_2 \cap A_1B_2$, $P_2 = B_1A_3 \cap A_1B_3$, $P_1 = B_3A_2 \cap A_3B_2$.
Then the points P_1, P_2, P_3 are collinear.

Remark 4.24. (Dual statement to Pappus' theorem)
Let A and B be points and a_1, a_2, a_3 be lines through A, b_1, b_2, b_3 be lines through B.
Let p_1 be a line through $b_2 \cap a_3$ and $a_2 \cap b_3$,
p_2 be a line through $b_1 \cap a_3$ and $a_1 \cap b_3$,
p_3 be a line through $b_2 \cap a_1$ and $a_2 \cap b_1$.
Then the lines p_1, p_2, p_3 be concurrent.
[This is actually the same statement as Pappus' theorem.]

Remark 4.25. Pappus' theorem is a special case of Pascal's Theorem:
If A, B, C, D, E, F lie on a conic then the points $AB \cap DE$, $BC \cap EF$, $CD \cap FA$ are collinear.
[Without proof.]

Theorem 4.26. (Deasargues' theorem). Suppose that the lines joining the corresponding vertices of triangles $A_1A_2A_3$ and $B_1B_2B_3$ intersect at one point S.
Then the intersection points $P_1 = A_2A_3 \cap B_2B_3$,
$P_2 = A_1A_3 \cap B_1B_3$, $P_3 = A_1A_2 \cap B_1B_2$ are collinear.

4.5 Hyperbolic geometry: Klein model

Model: in interior of unit disc.

- points - points; • lines - chords • distance: $d(A,B) = \frac{1}{2}\big|ln|[A,B,X,Y]|\big|$

where X, Y are the endpoints of the chord through AB and $[A,B,X,Y]$ is the cross-ratio.

Remark: 1. Axioms of Euclidean geometry are satisfied (except for Parallel Axiom).
2. Parallel axiom is obviously not satisfied:
Given a line l and a point $A \notin l$, there are infinitely many lines l' s.t. $A \in l$ and $l \cap l' = \emptyset$.

Theorem 4.27. The function $d(A,B)$ satisfies axioms of distance, i.e.
1) $d(A,B) \geq 0$ and $d(A,B) = 0 \Leftrightarrow A = B$;
2) $d(A,B) = d(B,A)$
3) $d(A,B) + d(B,C) \geq d(A,C)$.

Isometries of Klein model

Theorem 4.28. There exists a projective transformation of the plane that
- maps a given disc to itself,
- preserves cross-ratios of collinear points;
- maps the centre of the disc to an arbitrary inner point.

Corollary 4.29. Isometries act transitively on the points of Klein model.
Isometries act transitively on the flags in Klein model.

Remark. 1. In general, angles in Klein model are not represented by Euclidean angles.
2. Angles at the centre are Euclidean angles.
3. Right angles are shown nicely in the Klein model.

Proposition 4.30. Let l and l' be two lines in the Klein model.
Let t_1 and t_2 be tangent lines to the disc at the endpoints of l.
Then $l \perp l' \Leftrightarrow t_1 \cap t_2 \in l'$.

Pairs of lines in hyperbolic geometry: two lines in hyperbolic geometry are called
intersecting if they have a common point inside hyperbolic plane;
parallel if they have a common point on the boundary of hyperbolic plane;
divegent or ultraparallel otherwise.

Proposition-Exercise. Any pair of divergent lines has a unique common perpendicular.

Hierarchy of geometries

S^2 $\qquad$ $(S^2 \in \mathbb{E}^3, \quad S^2, \mathbb{E}^2, \mathbb{H}^2 \in \mathbb{H}^3)$
$\mathbb{E}^2 \subset Sim(2) \subset Aff(2) \subset \mathbb{RP}^2$
$\mathbb{H}^2 \qquad \subset \qquad \mathbb{RP}^2$

5 Möbius geometry

5.1 Group of Möbius transformations

Definition 5.1. A map $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ given by $f(z) = \frac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{C}$, $ad - bc \neq 0$ is called a Möbius transformation or a linear-fractional transformation.

Remark. It is a bijection of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ to itself .

Theorem 5.2. (a) Möbius transformations form a group (denote it $M\ddot{o}b$),
this group is isomorphic to $PGL(2,\mathbb{C}) = GL(2,\mathbb{C})/\{\lambda I \mid \lambda \neq 0\}$.
(b) This group is generated by $z \to az$, $z \to z+b$ and $z \to 1/z$.

Theorem 5.3. (a) Möbius transformations act on $\mathbb{C} \cup \{\infty\}$ triply-transitively.
(b) A Möbius transformation is uniquely determined by the images of 3 points.

Theorem 5.4 Möbius transformations (a) take lines and circles to lines and circles and
(b) preserve angles between curves.

5.2 Types of Möbius transformations

Definition 5.5. A Möbius transformation with a unique fixed point is called parabolic.

Proposition 5.6. Every parabolic Möbius transformation is conjugate in the group *Möb* to $z \to z+1$.

Proposition 5.7. Every non-parabolic Möbius transformation is conjugate in *Möb* to $z \to az$, $a \in \mathbb{C} \setminus \{0\}$.

Definition 5.8. A non-parabolic Möbius transformation conjugate to $z \to az$ is called
(1) elliptic, if $|a| = 1$; (2) hyperbolic, if $|a| \neq 1$ and $a \in \mathbb{R}$; (3) loxodromic, otherwise.

Remark. Two fixpoints of a hyperbolic or a loxodromic transformation have different properties: one is attracting another is repelling.
Elliptic transformations have two similar fixpoints (neither attracting nor repelling).

5.3 Inversion

Definition 5.9. Let $\gamma \in \mathbb{C}$ be a circle with centre O and radius r. An inversion I_γ with respect to γ takes a point A to a point A' lying on the ray OA s.t. $|OA| \cdot |OA'| = r^2$.

Proposition 5.10. (a) $I_\gamma^2 = id$. (b) Inversion in γ preserves γ pointwise ($I_\gamma(A) = A$ for all $A \in \gamma$).

Lemma 5.11. If $P' = I_\gamma(P)$ and $Q' = I_\gamma(Q)$ then $\triangle OPQ$ is similar to $\triangle OQ'P'$.

Theorem 5.12. Inversion takes circles and lines to circles and lines. More precisely,
1. lines through O $\leftrightarrow$ lines through O
2. lines not through O $\leftrightarrow$ circles through O
3. circles not through O $\leftrightarrow$ circles not through O

Theorem 5.13. Inversion preserves angles.

Remark. Inversion may be understood as "reflection with respect to a circle":

Corollary 5.14. Every inversion is conjugate to a reflection by another inversion.

Theorem 5.15. Every Möbius transformation is a composition of even number of inversions and reflections.

Remark. Inversion and inversion change orientation of the plane.
Theorem 5.15 shows that Möbius transformations preserve orientation.

5.4 Möbius transformations and cross-ratios

Definition 5.16. For $z_1, z_2, z_3, z_4 \in \mathbb{C} \cup \{\infty\}$,
the complex number $[z, z_2, z_3, z_4] = \frac{z_3 - z_1}{z_3 - z_2} / \frac{z_4 - z_1}{z_4 - z_2} \in \mathbb{C} \cup \{\infty\}$ is called the cross-ratio.

Theorem 5.17. Möbius transformations preserve cross-ratios.

Proposition 5.18. Points $z_1, z_2, z_3, z_4 \in \mathbb{C} \cup \{\infty\}$ lie on one line or circle iff $[z_1, z_2, z_3, z_4] \in \mathbb{R}$.

Proposition 5.19. Given four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C} \cup \infty$, one has $[z_1, z_2, z_3, z_4] \neq 1$.

Example. One can use cross-ratio to determine whether a given pair of disjoint circles is Möbius-equivalent to another given pair.

5.5 Inversion in space

Remark. Reflection and inversion take the cross-ratio to the complex-conjugate value.

Corollary 5.20. For points z_1, z_2, z_3, z_4 lying on a circle or on a line, inversions and reflections do preserve the cross-ratio.

Definition. Let $S \in \mathbb{R}^3$ be a circle with centre O and radius r. An inversion I_S with respect to S takes a point $A \in \mathbb{R}^3 \cup \{\infty\}$ to a point A' lying on the ray OA s.t. $|OA| \cdot |OA'| = r^2$.

Theorem 5.21. (Properties of inversion).
1. Inversion takes spheres and planes to spheres and planes.
2. Inversion takes lines and circles to lines and circles.
3. Inversion preserves angles between curves.
4. Inversion preserves cross-ratio of four points $[A, B, C, D] = \frac{|CA|}{|CB|} / \frac{|DA|}{|DB|}$.

5.6 Stereographic projection

Definition. Let S be a sphere centred at O, let Π be a plane through O. Let $N \in S$ be a point with $NO \perp \Pi$. The map $\pi : S \to \Pi$ s.t. $\pi(A) = \Pi \cap NA$ for all $A \in S$ is called a stereographic projection.

Proposition 5.22. (Properties of stereographic projection).
1. Stereographic projection takes circles to circles and lines.
2. Stereographic projection preserves the angles.
3. Stereographic projection preserves cross-ratio.

Remark. Another way to define stereographic projection, is to project from N to the plane tangent to S at point opposite to N. This projection has the same properties.

Example: Steiner Porism. A circle γ_1 lies inside another circle γ_2. A circle $\mathcal{C}_0$ is tangent to both γ_1 and γ_2. A circle $\mathcal{C}_i$ is tangent to three circles: γ_1, γ_2 and $\mathcal{C}_{i-1}$, for $i = 1, 2, 3 \dots$. It may happen that either all circles $\mathcal{C}_i$, $i \in \mathbb{N}$ are different, or $\mathcal{C}_n = \mathcal{C}_1$ for some n. Show that the outcome does not depend on the choice of the inicial circle $\mathcal{C}_0$ (but only depends on γ_1 and γ_2).
Hint: First show that every two disjoint circles are Möbius-equivalent to two concentric circles.

6 Hyperbolic geometry: conformal models

6.1 Poincaré disc model

Model: $\mathbb{H}^2$ =unit disc $D = \{|z| < 1, z \in \mathbb{C}\}$;
$\partial\mathbb{H}^2 = \{|z| = 1\}$, boundary, called absolute;
lines: parts of circles or lines orthogonal to $\partial\mathbb{H}^2$;
isometries: Möbius transformation, inversions, reflections - preserving the disc;
distance: a function of cross-ratio;
angles: same as Euclidean angles.

Proposition 6.1. For any two points $A, B, \in \mathbb{H}^2$ there exists a unique hyperbolic line through A, B.

Remark: The same holds for $A, B, \in \mathbb{H}^2 \cup \partial\mathbb{H}^2$.

Definition 6.2. $d(A, B) = \big|ln|[A, B, X, Y]|\big| = \big|ln\frac{|XA|}{|XB|}/\frac{|YA|}{|YB|}\big|$,
where X, Y are the points of the absolute contained in the (hyperbolic) line AB.

Theorem 6.3. $d(A, B)$ satisfies axioms of the distance.

Example: rotation about the centre of the model, reflection with respect to a diameter, inversion with respect to a circle representing a hyperbolic line are isometries of the Poincaré disc model.

Proposition 6.4. Let $l \in \mathbb{H}^2$ be a (hyperbolic) line, $A \in \mathbb{H}^2$ or $A \in \partial\mathbb{H}^{"}$ be a point, $A \notin l$.
Then there exists a unique line l' through A orthogonal to l.

Proposition 6.5. Let $l \in \mathbb{H}^2$ be a (hyperbolic) line, $A \in \mathbb{H}^2$ or $A \in \mathbb{H}^2$ be a point, $A \in l$.
Then there exists a unique line l' through A orthogonal to l.

Proposition 6.6. Every hyperbolic segment has a midpoint.

Remark. When $B = B(t)$ runs along a ray AX from A to X,
the distance $d(A, B(t))$ grows monotonically from 0 to ∞.

Theorem 6.7. The isometry group group of $\mathbb{H}^2$ acts transitively
(1) on triples of points of the absolute;
(2) on flags in $\mathbb{H}^2$.

Proposition 6.8. For $C \in AB$, $d(A, C) + d(C, B) = d(A, B)$.

Lemma 6.9. In a right-angled triangle ABC with $\angle A = \pi/2$, $d(A, B) < d(C, B)$.

Proposition 6.7. Hyperbolic circles are represented by Euclidean circles in the Poincaré disc model.

Corollary 6.10. Triangle inequality: for $C \notin AB$, $\quad d(A, C) + d(C, B) > d(A, B)$.

Remark. Triangle inequality implies that (a) distance is well-defined, and
(b) hyperbolic lines are geodesics in the model.

Lemma 6.11. Hyperbolic circles are represented by Euclidean circles in the Poincaré disc model.

Theorem 6.12. An isometry of $\mathbb{H}^2$ is uniquely determined by the image of a flag.

Theorem 6.13. Every isometry of the Poincaré disc model can be written as
either $\frac{az+b}{cz+d}$ (Möbius transformation) or $\frac{a\bar{z}+b}{c\bar{z}+d}$ (anti-Möbius transformation).

Corollary. An isometry of $\mathbb{H}^2$ is uniquely determined by the amages of three points of the absolute.

Corollary. Isometries preserve the angles.

Proposition 6.14. The sum of angles in a hyperbolic triangle is less than π.

Remark. If $\alpha + \beta + \gamma < \pi$ then there exists a triangle with angles α, β, γ.

6.2 Upper half-plane model

Model: $\mathbb{H}^2 = \{z \in \mathbb{C},\ Imz > 0\}$;
$\partial\mathbb{H}^2 = \{Imz = 0\}$, absolute;
lines: rays and half-circles orthogonal to $\partial\mathbb{H}^2$;
distance: $d(A, B) = \big|ln[A, B, X, Y]\big|$;
isometries: Möbius transformation, inversions, reflections - preserving the half-plane;
angles: same as Euclidean angles.

Proposition 6.15. This defines the same geometry as Poincaré disc model.

Proposition 6.16. In the upper half-plane, hyperbolic circles are represented by Euclidean circles.

Theorem 6.17. In the upper half-plane model, $\cosh d(z, w)) = 1 + \frac{|z-w|^2}{2Im(z)Im(w)}$.

Theorem 6.18. Every isometry of the upper half-plane model can be written as
either $z \mapsto \frac{az+b}{cz+d}$ or $z \mapsto \frac{a(-\bar{z})+b}{c(-\bar{z})+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$.

Equivalently, or.-preserving isometries can be written as $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$;
Hence, for the group of or.-preserving isometries we have $Isom^+(\mathbb{H}^2) = PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\pm I$.

Orientation-reversing isometries can be written as $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc = -1$.

Example. Let l, l' be parallel lines. Then $d(l, l') = 0$.
(where by distance betrween the sets α and β we mean $d(\alpha, \beta) = \inf\limits_{A\in\alpha, B\in\beta}(A, B)$).

6.3 Elementary hyperbolic geometry

Remark. 1. Triangle inequality implies that $d(A, B)$ satisfies axioms of distance and that hyperbolic lines are shortest paths.
2. In hyperbolic geometry, all Euclid's Axioms, except Parallel Axiom hold.
3. Parallel Axiom for hyperbolic geometry says that there are more than one disjoint from a given l through a given point $A \notin l$.

Definition 6.19. For a line l and a point $A \notin l$, an angle of parallelism $\varphi = \varphi(A, l)$ is the half-angle between the rays emanating from A and parallel to l.
Equivalently: drop a perpendicular AH to l, then $\varphi = \angle HAQ$, $Q \in l \cap \partial\mathbb{H}^2$.
Equivalently: a ray AX from A intersects l iff $\angle HAY \le \varphi$.

Proposition 6.20. For a line l and a point $A \notin l$, let $a = d(A, l)$ and φ be the angle of parallelism.
Then $\cosh a = \frac{1}{\sin\varphi}$.

Theorem 6.21. (Hyp. Pythagorean theorem).
In a triangle with a right angle γ, $\cosh c = \cosh a \cosh b$.

Lemma 6.22. In a triangle with a right angle γ holds
$$\sinh a = \sinh c \sinh\alpha \qquad \text{and} \qquad \tanh b = \tanh c \cos\alpha.$$

Theorem 6.23. (Law of sines) $\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}$.

Theorem 6.24. (Law of cosines) $\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha$.

Remark. For small values of a, b, c we get Euclidean sine and cosine laws.

Theorem 6.25. (Second Law of cosines) $\cos\alpha = -\cosh\beta\cos\gamma + \sin\beta\sin\gamma\cosh a$.

Exercise: prove SSS, SAS, ASA and AAA rules of congruence of triangles
(do it in two ways: with since/cosine law and without).

Example: Use sine law to compute length of circle of radius r: $l(r) = 2\pi\sinh r$.

Corollary. Uniform statement for sine law in S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$: $\frac{l(a)}{\sin\alpha} = \frac{l(b)}{\sin\beta} = \frac{l(c)}{\sin\gamma}$,
where $l(r)$ is the length of circle of radius r in the corresponding geometry.

Remark. In the hyperbolic geometry, the circle length $l(r)$ grows exponentially when $r \to \infty$.

Remark. Hyperbolic geometry is natural (salad leaf) and intuitive
("city model", refraction in the half-plane with density $1/y$).

6.4 Area of hyperbolic triangle

Theorem 6.26. $S_{\triangle ABC} = \pi - (\alpha + \beta + \gamma)$.

Corollary 6.27. Area of an n-gon: $S_n = (n-2)\pi - \sum_{i=1}^{n} \alpha_i$.

Example. Area of hyperbolic disc of radius r is $4\pi \sinh^2(\frac{r}{2})$.

7 Other models of hyperbolic geometry

7.1 Klein disc, revised

Reminder: lines are represented by chords, distance in Klein disc $d(A,B) = \frac{1}{2}\big|ln[A,B,X,Y]\big|$,
isometries are projective maps preserving the disc.

Theorem 7.1. Geometry of the Klein disc coincides with geometry of the Poincaré disc.

Remark: Hemisphere model can be projected to Klein disc, Poincare disc and upper half-plane.

Remark: When to use the Klein disc model? Working with lines and right angles.

Examples: perpendicular lines, common perpendicular to two ultra-parallel lines,
midpoint of a segment, angle bisector.

Remark: circles in the Klein model are represented by ellipses.

Remark: In the Poincare disc and in tre upper half-plane model,
every Euclidean circle represents some hyperbolic circle.

7.2 The model in two-sheet hyperboloid

Consider the hyperboloid $x_1^2 + x_2^2 - x_3^2 = -1$, $x_1, x_2, x_3 \in \mathbb{R}$; identify $(x_1, x_2, x_3) \sim (-x_1, -x_2, -x_3)$

Model: $\mathbb{H}^2 = \{$ points of the upper sheet$\}$ $\sim$ lines through O;
$\partial\mathbb{H}^2 = \{$(projectivised) points of the cone $x_1^2 + x_2^2 - x_3^2 = 0$ $\}$, $\sim$ lines spanning the cone ;
lines in H^2: intersections of planes through O with the hyperboloid;
distance: $d(A,B) = \frac{1}{2}\big|ln[A,B,X,Y]\big|$ cross-ratio of four lines in $\mathbb{R}^3$;
isometries: projective transformations preserving the cone.

Theorem 7.2. This determines the same hyperbolic geometry as the Klein model.

For $x = (x_1, x_2, x_3)$, $y = (y_1, Y_2, y_3)$ define a pseudo-scalar product $(x, y) = x_1y_1 + x_2y_2 - x_3y_3$.

Then • points of the $\mathbb{H}^2$: $(x, x) = -1$;
• points of the $\partial\mathbb{H}^2$: $(x, x) = 0$;
• hyperbolic line l_a: $a_1x_1 + a_2x_2 - a_3x_3 = 0$, i.e. $(a, x) = 0$.

Remark. if $(a, a) > 0$ then l_a intersects the cone and give a hyperbolic line;
if $(a, a) = 0$ then l_a is tangent to cone and give the point a on the absolute;
if $(a, a) < 0$ then l_a does not intersect the cone and give no line (but a give a point of $\mathbb{H}^2$).

Theorem 7.3. $\cosh^2 d(u, v) = \frac{(u,v)^2}{(u,u)(v,v)}$ for $u, v \in \mathbb{H}^2$, i.e. for u, v satisfying $(u, u) < 0$, $(v, v) < 0$.

Theorem 7.3. More distance formulae in terms of $Q = |\frac{(u,v)^2}{(u,u)(v,v)}|$:

- if $(u, u) < 0$, $(v, v) > 0$, then u gives a point and v give a line l_v on $\mathbb{H}^2$, and $\sinh^2 d(u, l_v) = Q$;
- if $(u, u) > 0$, $(v, v) > 0$ then u and v define two lines l_u and l_v on $\mathbb{H}^2$ and
  - ∘ if $Q < 1$, then l_u intersects l_v forming angle φ satisfying $Q = \cos^2 \varphi$;
  - ∘ if $Q = 1$, then l_u is parallel to l_v;
  - ∘ if $Q > 1$, then l_u and l_v are ultra-parallel lines satisfying $Q = \cosh^2 d(l_u, l_v)$.

8 Classification of isometries

8.1 Reflections

Definition. A reflection r_l with respect to a hyperbolic line l is an isometry preserving the line l pointwise and swapping the half-planes.

Example. • In the Poincaré disc and upper half-plane models:
reflections are represented by Euclidean reflections and inversions.
• In the Klein disc model: geven A and l, one can construct $r_l(A)$.

Theorem 8.1. In hyperboloid model: given a s.t. $(a,a) > 0$ (i.e. $(x,a) = 0$ defines a line l_a), the map $r_a : x \mapsto x - 2\frac{(x,a)}{(a,a)}a$ is the reflection with respect to the line l_a.

8.2 Classification

Theorem 8.2. Any isometry of $\mathbb{H}^2$ is a composition of at most 3 reflections.

Corollary 8.3. A non-trivial orientation-preserving isometry of $\mathbb{H}^2$ has either 1 fixed point in $\mathbb{H}^2$, or 1 fixed point on the absolute, or two fixed points on the absolute.

Definition 8.4. A non-trivial orientation-preserving isometry of $\mathbb{H}^2$ is called
elliptic if it has 1 fixpoint in $\mathbb{H}^2$,
parabolic if it has 1 fixpoint in $\partial\mathbb{H}^2$,
hyperbolic if it has 2 fixpoints in $\partial\mathbb{H}^2$.

Example. In the upper half-plane model, the transformation $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ is elliptic if $|d + a| < 2$, parabolic if $|d + a| = 2$ and hyperbolic if $|d + a| > 2$.

Observation 8.6: Invariant sets for isometries (sets preserved by a given isometry) for elliptic, parabolic and hyperbolic isometries are circles, horocycles and equidistant curves.

8.3 Horocycles and Equidistant curves.

Definition. A *circle* is a set of points on the same distance from a given point (centre).

Properties: 1. All lines through the centre are orthogonal to the circle.
2. The distance between two concentric circles γ and γ' is constant (i.e. given a point $A \in \gamma$ and a closest to A point $A' \in \gamma'$, the distance $d(A, A')$ does not depend on the choice of A).

Definition 8.7. A horocycle h is a limit of circles:
let $P \in \mathbb{H}^2$ be a point, and l be a ray from P; for $t > 0$ let $O_t \in l$ be a point s.t. $d(P, O_t) = t$; let $\gamma(t)$ be a circle centred at O_t of radius t; then a horocycle $h = \lim\limits_{t\to\infty} \gamma(t)$.
The point $X = \lim\limits_{t\to\infty} O(t) \in \partial\mathbb{H}^2$ is called the centre of the horocycle h.

Remark. In the Poincaré disc, every circle tangent to the absolute represents some horocycle.

Properties: 1. All lines through the centre of the horocycle are orthogonal to the horocycle.
2. The distance between two concentric horocycles h and h' is constant. (i.e. given a point $A \in h$ and a closest to A point $A' \in h'$, the distance $d(A, A')$ does not depend on the choice of A).

Definition 8.8. An equidistant curve e to a line l is a locus of points on a given distance from l.

Examples. In UHP, if l is a vertical ray 0∞, then e is a union of to (Euclidean) rays from 0 making the same angle with l. If l is a half of Euclidean circle, then e is a “banana”. In the Poincaré disc: also get banana.

Properties: 1. All lines orthogonal to l are orthogonal to the equidistant curve.
2. Two equidistant curves to the same line stay on the same distance.

Remark. 1. For elliptic, parabolic and hyperbolic isometry f of $\mathbb{H}^2$, through each point of $\mathbb{H}^2$ there is a unique invariant curve of the f (circle, horocycle or equidistant curve) and a unique line orthogonal to all invariant curves.
2. Representation of elliptic, parabolic and hyperbolic isometries as $r_2 \circ r_1$ is not unique: r_1 is a reflection with respect to any line from the orthogonal family, then there is a unique choice for r_2.

9 Geometry in modern mathematics (some topics) (NON-Examinable Section!!!)

9.1 Taming infinity via horocycles

Martin Hairer (2014 Fields medallist):
"Renormalisation" $\approx$ "If you have a diverging integral, substruct infinity (in a coherent way) and work with finite values"

We illustrate this with horocycles:
- any point of a horocycle h is on infinite distance from the centre X of the horocycle;
- two concentric horocycles are on a finite distance from each other;
- choose "level zero" horocycle, and measure the (signed) distance to it.

Lambda-length: - Given $X, Y \in \partial H^2$, choose horocycles h_X and h_Y centred at these points.
- Let l_{XY} be the finite portion of the line XY lying outside of both h_X and h_Y (it is a signed length, may be zero or negative if h_x intersects h_y).
- Define $\lambda_{XY} = exp(l_{XY}/2)$.

Ptolemy Thm. In $\mathbb{E}^2$, a cyclic quadrilateral $ABCD$ satisfies $|AC| \cdot |BD| = |AB| \cdot |CD| + |AD| \cdot |BC|$.

Hyperbolic Ptolemy Thm. For an ideal quadrilateral $ABCD$ (i.e. $A, B, C, D, \in \partial\mathbb{H}^2$), choose any horocycles centred at A, B, C, D. Then $\lambda_{AC} \cdot \lambda_{BD} = \lambda_{AB} \cdot \lambda_{CD} + \lambda_{AD} \cdot \lambda_{BC}$.

Remark. 1. The identity does not depend on the choice of the horocycles: if we change one horocycle taking another horocycles on distance d, all summands of the identity will be multiplies by $exp(d/2)$.
2. The proof of the Hyp. Ptolemy Thm. is an elementary computation in the UHP - omitted.

Lemma. Given an ideal triangle A_1, A_2, A_3 and $c_{12}, c_{23}, c_{31} \in \mathbb{R}_{\geq 0}$ there exists a unique choice of horocycles centred at A_1, A_2, A_3 such that $\lambda_{A_iA_j} = c_{ij}$.

Remark. This allows: - to introduce and study hyperbolic structures on triangulated surfaces;
- to define an important class of *cluster algebras*.

9.2 Three metric geometries: S^2, $\mathbb{E}^2$, $\mathbb{H}^2$, unified

S^2: $d(A, B) = r\varphi$. (is tending to $\mathbb{E}^2$ when $r \to \infty$).
$\mathbb{H}^2$: $d(A, B) = R|ln[A, B, X, Y]|$ (is tending to $\mathbb{E}^2$ when $R \to \infty$).

Remark. We use complex projective geometry to show that $d(A, B) = \pm\frac{r}{2i}|ln[A, B, X, Y]|$ for S^2.
(In case of $\mathbb{H}^2$ we consider the hyperboloid as a sphere $x_1^2 + x_2^2 + x_3^2 = -R^2$ of imaginary radius iR, rewriting this for $x_3' = ix_3$ we get exactly the hyperboloid model.)
To find the points X, Y we use the same rule as in the hyperboloid model: $\{X, Y\} = \Pi_{AB} \cap \{(x, x) = 0\}$.
- Here, the plane through $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ is $(a_1 + \lambda b_1, a_2 + \lambda b_2, a_3 + \lambda b_3)$.
- Intersection with the cone $(x, x) = 0$ gives $(a_1 + \lambda b_1)^2 + (a_2 + \lambda b_2)^2 + (a_3 + \lambda b_3)^2 = 0$.
- Taking in account $(a, a) = r^2 = (b, b)$ and $(a, b) = r^2 \cos\varphi$ this gives $1 + 2\lambda cos\varphi + \lambda^2 = 0$.
- Solving for λ we get x and y: $\lambda_{1,2} = -cos\varphi \pm i \sin\varphi$;
- $[a, b, x, y] = [0, \infty, \lambda_1, \lambda_2] = exp(\mp 2i\varphi)$, i.e. $\varphi = \mp\frac{r}{2i}|ln[a, b, x, y]|$, and $d(A, B) = \pm\frac{r}{2i}|ln[A, B, X, Y]|$.

Remark. This explains appearence of similar formulae in spherical and hyperbolic geometries, in particular, this gives a proof of the second cosine law in the hyperbolic case.

Comparison Theorem (Aleksandrov-Toponogov). Given $a, b, c \in R_{\geq 0}$ such that $a + b < c$, $a + c < b$ and $b + c < a$, consider triangles in $\mathbb{H}^2, \mathbb{E}^2$ and S^2 with sides a, b, c. Let $m_{\mathbb{H}^2}$, $m_{\mathbb{E}^2}$ and m_{S^2} be the medians connecting C with the midpoint of AB in each of the three triangles. Then $m_{\mathbb{H}^2} < m_{\mathbb{E}^2} < m_{S^2}$.

9.3 Discrete groups of isometries of $\mathbb{H}^2$: Examples

Idea: Tessellation by polygons (copies of F) $\to$
side pairings ($\forall a_i \in F$ there is $g_i : a_i \in g_F$) $\to$
oriented graph Γ: vertices of Γ $\leftrightarrow$ vertices A_i of F,
edges of γ $\leftrightarrow$ side pairings: A_i–A_j if $g_i(A_i) = A_j$ $\to$
Γ is a union of cycles, vertices in one cycles are called equivalent.

Lemma. Let $A_1, \dots, A_k$ make on cycle, so that $g_i(A_i) = A_{i+1}$, $g_k(A_k) = A_1$, where g_i are side pairings of F and $A_1, \dots, A_k \in \mathbb{H}^2$ (but not $\partial\mathbb{H}^2$). then $g = g_k g_{k_1} \dots g_1$ is a rotation about A_1 by the angle $\alpha_1 + \dots + \alpha_k$, where α_i is the angle of F at A_i.

Claim. Polygons g_kF, $g_kg_{k-1}F$,...,$g_kg_{k-1} \dots g_1F$ have a common vertex A_1, with angles $\alpha_k, \alpha_{k-1}, \dots, \alpha_1$ at A_1.

Corollary. Elements of the group $< g_1, \dots, g_n >$ generated by side pairings tile the neighbourhood of A_1 iff $\alpha_1 + \dots \alpha_k = 2\pi/m$ for $m \in \mathbb{N}$.

This necessary condition is also sufficient:

Poincaré's Theorem. Let $F \subset \mathbb{H}^2$ be a convex polygon, finite sided, no ideal vertices, s.t.
a) it's sides are paired by orientation preserving isometries $\{g_1, \cdot, g_n\}$;
b) angle sum in equivalent vertices is $2\pi/m_i$ for $m_i \in \mathbb{N}$.
Then 1) the group $G =< g_1, \cdot, g_n$ is discrete;
2) F is its fundamental domain.

Examples: the following groups are discrete (more examples than we had in the lecture):

1a. P= regular hexagon in $\mathbb{E}^2$, G generated by translations pairing the opposite sides of P.

1b. P= regular hexagon in $\mathbb{E}^2$, G gen. by rotations by $2\pi/3$ about three non-adjacent vertices.

2. Regular hyperbolic octagon with angles $\pi/(4m)$, $m \in \mathbb{Z}$,
   G generated by translations pairing the opposite sides of P.
3. P polygon all whose angles are integer submultiples of π, i.e. π/m (called Coxeter polygons),
   G generated by reflections with respect to the sides of P.

9.4 Hyperbolic surfaces

Definition. A surface S is called hyperbolic if every point $p \in S$ has a neighbourhood isometric to a disc on $\mathbb{H}^2$.

How to construct?

1. Glue from hyperbolic polygons.

Examples: Euclidean torus glued from a square with identified opposite sides;
Hyperbolic surface of genus 2 ("two holed torus")
glued of a regular octagon with angles $\pi/4$ (opposite sides identified).

2. Pants decompositions.

A pair of pants is a sphere with three holes.
A hyperbolic pair of pants may be glued from to right-angled hyperbolic hexagons. Glueing several pairs of pants by the boundaries, one can get (almost) every compact topological surface.
Exceptions are a sphere and a torus, which naturally carry spherical and Euclidean geometry, but not hyperbolic.

3. Quotient of $\mathbb{H}^2$ by a discrete group.

Let $G : \mathbb{H}^2$ be a discrete action. Consider an orbit space $\mathbb{H}^2/G$.
Sometimes we get a hyperbolic surface, but not always.

Example. A regular hyperbolic quadrilateral with angles $\pi/4$ and opposite sides identified
gives a torus with a cone point (angle π around the image of the vertices).
It is not a manifold (this structure is called an orbifold).

4. Developing map.

For each loop on a surface we construct a path on $\mathbb{H}^2$. So, each loop on S give rise to an isometry of $\mathbb{H}^2$.
Consider a group G generated by all these isometries.
G acts on $\mathbb{H}^2$ discretely , and $S = \mathbb{H}^2/G$ is its orbit space.
Also, $\forall g \in G, x \in \mathbb{H}^2$ if $gx = x$ then $g = id$ (such an action $G : X$ is called free).

5. Uniformisation theorem. A closed oriented hyperbolic (or Euclidean,,or spherical) surface is a quotient of $\mathbb{H}^2$ (or $\mathbb{E}^2$, or S^2) by a free action of a discrete group.

9.5 Review via 3D

I. Four models of $\mathbb{H}^3$

Ia. Upper half-space. Space: $\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 \mid t > 0\}$.
Absolute: $\partial\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 \mid t = 0\}$.
Hyperbolic lines: vertical rays and half-circles orthogonal to the absolute.
Hyperbolic planes: vertical (Euclidean) half-planes and half-spheres centred at the absolute.

$d(A,B) = |ln[A,B,X,Y]|$ (X,Y the ends of the line, cross-ratio computed in a vertical plane).
$\cosh d(u,v) = 1 + \frac{|u-v|^2}{2u_3v_3}$.
Isometries.
Example: Hyperbolic reflections= (Eucl) reflections with respect to the vertical planes and
Inversions with respect to the spheres centred at the absolute.
- $f \in Isom(\mathbb{H}^3)$ is determined by it's restriction to the absolute.
- $Isom(\mathbb{H}^3)$ is generated by reflections (every isometry is a composition of at most 4).
- Restrictions to $\partial\mathbb{H}^3$ are compositions of (Eucl.) reflections and inversions.
- $Isom^+\mathbb{H}^3 = M\ddot{o}b$.

Spheres: Euclidean spheres (another centre).
Horospheres (limits of spheres): horizontal planes and spheres tangent to the absolute.
Equidistant (to a line): vertical cone (or banana for "half-circle" lines).
Equidistant (to a plane Π): two (Eucl) planes at the same angle to a vertical plane (at $\Pi \cap \partial\mathbb{H}^3$)
or two pieces of spheres at the same angle to the sphere representing Π.

Ib. Poincaré ball.
Obtained by inversion from the upper half-space model.
Space: $\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 \mid x^2+y^2+t^2 < 1\}$.
Absolute: $\partial\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 \mid x^2+y^2+t^2 = 1\}$.
Hyperbolic lines: parts of lines and circles orthogonal to $\partial\mathbb{H}^3$.
Hyperbolic planes: parts of planes and spheres orthogonal to $\partial\mathbb{H}^3$.
$d(A,B) = |ln[A,B,X,Y]|$ (X,Y the ends of the line, cross-ratio computed in a plane).

Both Poincaré models are conformal: hyperb. angles are represented by Eucl. angles of the same size.

Ic. Klein model.
Space: $\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 \mid x^2+y^2+t^2 < 1\}$.
Absolute: $\partial\mathbb{H}^3 = \{(x,y,t) \in \mathbb{R}^3 \mid x^2+y^2+t^2 = 1\}$.
Hyperbolic lines: chords.
Hyperbolic planes: intersections with Euclidean planes.
$d(A,B) = \frac{1}{2}|ln[A,B,X,Y]|$ (X,Y the ends of the line).
Angles are distorted (except ones at the centre).
Right angles are easy to control.

Id. Hyperboloid model.
Hyperboloid: $x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1$, $x \in \mathbb{R}^4$.
Pseudo-scalar product: $(x,y) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$.
Space: $(x,x) = -1$.
Absolute: $(x,x) = 0$.
Hyperbolic planes: $(x,a) = 0$ for a s.t. $(a,a) > 0$.
$d(A,B) = \frac{1}{2}|ln[A,B,X,Y]|$ (cross-ratio of four lines).
$cosh^2(d(pt_1,pt_2) = Q(pt_1,pt_2)$ where $Q(u,v) = \frac{(u,v)^2}{(u,u)(v,v)}$.

II. Orientation-preserving isometries of $\mathbb{H}^3$

- In the upper half-space, or.-preserving isometries correspond to Möbius transformation of $\partial\mathbb{H}^3$:
$\frac{az+b}{cz+d}$ with $z \in \partial\mathbb{H}^3$, $a,b,c,d \in \mathbb{C}$, $ad-bc \neq 0$.
- Parabolic: 1 fixpt on $\partial\mathbb{H}^3$, conjugate to $z \mapsto z+a$.
- Non-parabolic: 2fixpts on $\partial\mathbb{H}^3$, conjugate to $z \mapsto az$.
elliptic, $|a| = 1$, rotation about a vertical line.
hyperbolic, $a \in \mathbb{R}$, (Eucl) dilation.
loxodromix, (otherwise), "spiral trajectory" =composition of rotation and dilation

III. Some polytopes in $\mathbb{H}^3$

1. Ideal tetrahedron. It is not unique up to isometry! (There are 2 parameters=2 dihedral angles).
2. Regular right-angled dodecahedron.
3. Right-angled ideal octahedron.

IV. Geometric structures on 3-manifolds

- Can glue from polytopes.
- Need to check angles around edges and vertices.

V. Geometrisation conjecture

William Thurston: all topological 3-manifolds are geometric manifolds, i.e.

Every oriented compact 3-manifold without boundary can be cut into pieces having one of the following 8 geometries: S^3, $\mathbb{E}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, Sol and universal cover of $SL(2, \mathbb{R})$.

(1982) William Thurston: proved geometrisation conjecture for Haken manifolds. (Fields medal, 1982)
In particular, closed atoroidal Haken manifolds are hyperbolic.

(2003) Grigori Perelman: general proof of the geometrisation conjecture. (Fields medal, 2006)
This also proves Poincaré conjecture:
Every simply-connected closed 3-manifold is a 3-sphere. (Clay Millennium Prize).