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Questions from Problems classes

Problems Class 1 (24 October 2018)

Reflections on the plane, Geometric constructions

1. Example of using reflections to study compositions of isometries (write everything as a composition of reflections, make you choice so that some of them cancel!).
2. Example of using reflection to find a shortest way from a point A to a river and then to a point B on the same bank.
3. Ruler and compass constructions: perpendicular bisector, perpendicular from a point to a line, midpoint of a segment, angle bisector, inscribed and circumscribed circles for a triangle.

Problems Class 2 (7 November 2018)

Group actions on $\mathbb{E}^2$

0. Let $g_1, \dots, g_n$ be isometries of $\mathbb{E}^2$. Let $G = \langle g_1, \dots, g_n \rangle$ be the group generated by $g_1, \dots, g_n$ (i.e. the minimal group containing all of $g_1, \dots, g_n$). Show that the group G acts on $\mathbb{E}^2$.
1. Let G be a group generated by two reflections on $\mathbb{E}^2$. When G is discrete?
2. Let T be a triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$. Let r_1, r_2, r_3 be the reflection with respect to the sides of T, and let G be the group generated by r_1, r_2, r_3. In the lecture we have checked that $G : \mathbb{E}^2$ discretely. Find the fundamental domain of this action.
3. Find the orbit-space for the action introduced in Question 2.
4. Let X be a regular triangle on $\mathbb{E}^2$. Let r_1 and r_2 be two distinct reflections taking X to itself. Find the fundamental domain of the action $G : X$. Find also the orbit-space.
5. Let G be a group generated by rotation through angle $\frac{2\pi}{3}$ on the plane. Find the orbit-space of the action $G : \mathbb{E}^2$. Are there closed geodesics in this orbit-space? Are there bi-infinite open geodesics?

Problems Class 3 (21 November 2018)

Some questions on the sphere S^2 and in affine geometry

1. Let T be a regular tetrahedron in $\mathbb{E}^3$.
   (a) Find the order of the group $G := Isom(T)$ - group of isometries of T.
   (b) Let S^2 be a sphere circumscibed around T. Show that G acts on S^2 discretely.
   (c) Find the fundamental domain for the action $G : S^2$.
2. Let S^2 be a sphere of radius R. Let α and β be two parallel planes crossing S^2. Find the area of the part of S^2 lying between the planes α and β.
3. Let S^2 be a sphere of radius 1. Show that the length of a circle or(spherical) radius r equals to $2\pi \sin r$.

   Remark: for the sphere of radius R, the length of the circle of radius r will be $2\pi R \sin(\frac{r}{R})$. When $R \to \infty$ we see that $\frac{r}{R} \to 0$ and, hence, $2\pi R \sin(\frac{r}{R}) \to 2\pi r$.
4. One can also discuss ruler and compass constructions, as in $\mathbb{E}^2$.
5. Use affine geometry to show, that three medians of a triangle on $\mathbb{E}^2$ are concurrent.

Problems Class 4 (5 December 2018)

Projective geometry

1. Find a projective transformation f which takes

$$\begin{aligned} A = (1:0:0) &\text{ to } (0:0:1) \\ A = (0:1:0) &\text{ to } (0:1:1) \\ A = (0:0:1) &\text{ to } (1:0:1) \\ A = (1:1:1) &\text{ to } (1:1:1) \end{aligned}$$

   Find the image of $X = AD \cap BC$ under this transformation.
2. Find $[A, B, C, D]$ for the points above. (Does it exist?)
   For $E = (1:1:0)$, $F = (1:2:0)$ find $[A, B, E, F]$.
3. Check explicitely, that the transformation f from Question 1 preserves the value of $[A, B, E, F]$.
4. Let A_1, A_2, A_3, A_4 be points on a line a, let B_1, B_2, B_3, B_4 be points on a line b. Denote by p_i the line through A_i and B_i. Show that if the lines p_1, p_2, p_3, p_4 are concurrent, then the points $A_{i+1}B_i \cap A_iB_{i+1}$ $(i = 1, 2, 3)$ are colliner.
5. Formulate and prove the statement dual to the one in Question 4.