

Geometry III/IV

Time and place: Tuesday 10:00, CM101
Wednesday (weeks 3,5,7,9) 10:00, CM101
Friday 16:00, CG60

Lecturer: Anna Felikson
e-mail: anna.felikson@durham.ac.uk

Office: CM124 Phone: 334-4158
Office hours: Tuesday 12:30-13:30

Course webpage: <http://www.maths.dur.ac.uk/users/anna.felikson/Geometry/Geometry18/index18.html>

Euclidean Geometry

EUCLID'S POSTULATES (5)

1. For every point A and for every point B not equal to A there exists a unique line that passes through A and B .
2. For every segment AB and for every segment CD there exists a unique point E such that B is between A and E and such that segment CD is congruent to segment BE .
3. For every point O and every point A not equal to O , there exists a circle with center O and radius OA .
4. All right angles are congruent to each other.
5. (Euclid's Parallel Postulate) For every line l and for every point P that does not lie on l , there exists a unique line m passing through P that is parallel to l .

HILBERT'S AXIOMS (5 groups)

Undefined notions: point, line, incidence, betweenness, and congruence.

1. Incidence Axioms (IA)

- IA 1. Given 2 distinct points there is a unique line incident with them.
- IA 2. Given a line there exist at least 2 distinct points incident with it.
- IA 3. There exist 3 distinct points not incident with the same line.

2. Betweenness Axioms (BA)

- BA 1. If $A * B * C$ then also $C * B * A$ and A, B, C are distinct collinear points.
- BA 2. Given 2 points P and Q there exist 3 points A, B, C such that $P * B * Q$ and $P * Q * C$ and $A * P * Q$.
- BA 3. Given 3 collinear points, only one of them can be between the other two.
- BA 4. (Plane Separation) For every line l and for every 3 points A, B, C not on l ,
 - (a) If A, B are on the same side of l and B, C are on the same side of l , then A, C are on the same side of l .
 - (b) If A, B are on the opposite sides of l and B, C are on the opposite sides of l , then A, C are on the same side of l .

3. Congruence Axioms (CA)

- CA 1. Given segment AB and any ray with vertex C , there is a unique point D on this ray such that $AB \approx CD$.
- CA 2. If $AB \approx CD$ and $AB \approx EF$ then $CD \approx EF$.
- CA 3. Given $A * B * C$ and $A' * B' * C'$, if $AB \approx A'B'$ and $BC \approx B'C'$ then $AC \approx A'C'$.
- CA 4. Given $\angle D$ and any ray AB there is a unique ray AC on each half-plane of the line AB such that $\angle BAC \approx \angle D$.
- CA 5. If $\angle A \approx \angle B$ and $\angle A \approx \angle C$ then $\angle B \approx \angle C$.
- CA 6. (SAS Criterion) If 2 sides and the included angle of a triangle are congruent to those of another triangle, respectively, then the two triangles are congruent.

4. Continuity Axioms (CtA)

- CtA 1. (Circular Continuity Principle) If a circle has one point inside and one point outside another circle, then the two circles intersect in two points.
- CtA 2. (Archimedes' Axiom) Given segment CD and any ray AB there is a number n and a point E on this ray such that $n \times CD \approx AE \geq AB$.

5. Parallelism Axiom (PA)

- PA 1. (Hilbert's Parallel Axiom) Given a line l and a point P not on l , there is at most one line through P which is parallel to l .

(SOME) THEOREMS of EUCLIDEAN GEOMETRY (E)

	Statement	Pf based on	reference	PA
	Basic facts:			
E1	Given a triangle $\triangle ABC$ and a line l s.t. $A, B, C \notin l$, if l intersects AB then l intersects either AC or BC .	BA4		-
E2	An angle and its supplement add to π . Corollary: Supplements of congruent angles are congruent.			-
E3	Vertical angles are equal.	E2		-
	Congruence of triangles, isosceles triangles:			
E4	(SAS) If $AB = A'B'$, $AC = A'C'$ and $\angle BAC = \angle B'A'C'$ then $\triangle ABC$ is congruent to $\triangle A'B'C'$.	=CA6	Axiom	-
E5	(ASA) If $AB = A'B'$, $\angle BAC = \angle B'A'C'$ and $\angle ABC = \angle A'B'C'$ then $\triangle ABC$ is congruent to $\triangle A'B'C'$.	CA1, CA4	Thm 1.2	-
E6	(Thales's thm) If $AB = BC$ then $\angle BAC = \angle BCA$.	E4 (SAS)	Ex. 1a.3	-
E7	If $\angle BAC = \angle BCA$ then $AB = BC$.	E5 (ASA)	Ex. 1b.1	-
E8	If $AB = BC$ and M is a midpoint of AC then BM bisects $\angle ABC$ and is orthogonal to AC . ("in isosceles triangle a median is an altitude and an angle bisector").	E4 (SAS)	Ex. 1a.4	-
E9	Given a line l and a point A there exists a unique line l' perpendicular to l and containing A .	CA4, Cor. E2		-
E10	(SSS) If $AB = A'B'$, $AC = A'C'$ and $BC = B'C'$ then $\triangle ABC$ is congruent to $\triangle A'B'C'$.	E9, E4 (SAS) ,E9	Thm 1.1.	-
	Parallel lines:			
E11	For lines a, b, c , if $a \parallel b$ and $b \parallel c$ then $a \parallel c$.	PA		+
E12	Distinct lines l and l' are parallel iff some transversal line creates included interior angles adding to exactly π . Corollary: $l \parallel l'$ iff alternate interior angles are equal.	BA4, E5 (ASA), E2	Thm 1.4	+
E13	Angle sum of any triangle equals to π .	E12, BA3, BA4	Ex. 1c.2	+
	Lines in triangles :			
E14	The locus of points on the same distance from A and B is a line, this line coincides with the perpendicular bisector to AB .	E8, E4(SAS)		-
E15	The perpendicular bisectors of the sides of a triangle are concurrent. Corollary: Each triangle has a circumscribed circle.	E14	Ex. 1g.3	-
E16	The angle bisectors in a triangle are concurrent. Corollary: Each triangle has an inscribed circle.	E5 (ASA)	Thm 4.7	-
E17	(Orthocentre property) The altitudes in a triangle are concurrent.	E12, E15	Thm. 4.5	+

	Similarity:			
E18	In a parallelogram opposite sides are equal.	E12, E5 (ASA)	Ex.1d.1	+
E19	Let $\angle BOA$ be an angle, let $A_1, A_2, A_3 \in OA$, $B_1, B_2, B_3 \in OB$ be points satisfying $A_1B_1 \parallel A_2B_2 \parallel A_3B_3$. If $A_1A_2 = A_2A_3$ then $B_1B_2 = B_2B_3$.	E18, E5 (ASA)		+
E20	If ABC is a triangle and D, E are points on AB, AC s.t. $DE \parallel BC$ then $AD : AB = AE : AC$. Converse thm also holds: if $AD : AB = AE : AC$ then $DE \parallel BC$.	E19	Thm 2.2, 2.3	+
E21	The medians of a triangle are concurrent (at a point G called centroid) and $AG = \frac{2}{3}AM$ for a median AM of $\triangle ABC$.	E19	Thm 2.7	+
E22	Triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar (i.e. $\angle A_i = \angle B_i$) if and only if then they are proportional (i.e. $\frac{A_1A_2}{B_1B_2} = \frac{A_2A_3}{B_2B_3} = \frac{A_1A_3}{B_1B_3}$).	E20	Thm 2.6	+
E23	Let CD be an altitude of a right triangle ABC ($\angle C = \frac{\pi}{2}$) then $AD \cdot DB = CD^2$.	E22		+
E24	(Pythagoras's thm) In a right triangle ABC ($\angle C = \frac{\pi}{2}$) holds $AB^2 = AC^2 + BC^2$.	E23	Thm. 1.8	+
E25	(Triangle inequality) $AB + BC \geq AC$.	E24		+
	Angles in a circle:			
E26	(Angles in a semicircle) Let AB be a diameter of a circle \mathcal{C} , and let $P \in \mathcal{C}$ be point ($P \neq A, B$). Then $\angle APB = \frac{\pi}{2}$.	E8, E13	Thm. 3.1	+
E27	(Angle at centre) Let AB be an arc of a circle \mathcal{C} , centre O , and let $P \in \mathcal{C}$ be a point contained in the same halfplane as O with respect to the line AB . Then $\angle AOB = 2\angle APB$.	E7, E13	Thm. 3.2	+
E28	(Angle in the same segment are equal) Let AB be an arc of a circle \mathcal{C} , and let $P_1, P_2 \in \mathcal{C}$ be two points in the same halfplane as O with resp. to AB . Then $\angle AP_1B = \angle AP_2B$.	E27	Thm. 3.3	+
E29	(Opposite angles of a cyclic quadrilateral add to π) If $ABCD$ is a quadrilateral with vertices on a circle then $\angle ABC + \angle BCD = \pi$.	E27, E13	Thm. 3.4	+
	Sine and cosine rules:			
E30	(Sine rule) $\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$	Def. of sin	Thm. 2.8	+
E31	(Cosine rule) $c^2 = a^2 + b^2 - 2ab \cos \gamma$	E24	Thm. 2.9	+

Remarks:

- References are given to A. D. Gardiner, C.J. Bradley, *Plane Euclidean Geometry*, UKMT, Leeds 2012.
- The third column contains hints to (one of the many possible!) proofs.
(In many cases we choose proofs different from ones in Gardiner, Bradley).
- Last column indicates use of the parallel axiom (PA) in the proof.
Some statement marked "+" are still valid in the absence of PA!
- For the detailed treatment of axiomatic foundations of Euclidean geometry see
M. J. Greenberg, *Euclidean and Non-Euclidean Geometries*, San Francisco: W. H. Freeman, 2008.